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TABLE OF CONTENTS

Varia 99

JOSEPH ALMOG/ Representation of the Fuzziness of the Concept of Relevant Implication Within a Multi-valued Model Theory 100

LEE G. ARCHIE/ A Simple Defense of Material Implication 119

FREDERICK A. JOHNSON/ A Three-valued Interpretation for a Relevance Logic 123

Symposium/ The P-W Problem 129

LAWRENCE POWERS/ On P-W 131

LAWRENCE POWERS/ Latest on a Logic Problem 143

NUEL D. BELNAP, JR./ The Two-property 173

J. A. KALMAN/ Computer Studies of $T_{\rightarrow}$-W-I 181

ROBERT K. MEYER/ A General Gentzen System for Implicational Calculi 189

Addresses of Contributors 202

VARIA

With this issue, we complete volume one of the Newsletter. We would like to express our gratitude to the Office of Research and Projects, Graduate School, Southern Illinois University, Edwardsville, and to its director, John Rodman, for its generous financial support which has allowed us to put as much into this Newsletter as we have.

We have enough material on hand at present for all of the first two issues of volume two and part of the third issue. The January 1977 issue will, we think, be out on schedule. Beginning with that issue, we are gradually switching over to a single-spaced format for papers, in order to save paper and reduce overall costs. We would appreciate future papers typed single-spaced, though for a while we will also be publishing some double-spaced articles.

Three of the papers in this issue (the two by Larry Powers and the one by Bob Meyer) are papers that have circulated privately for several years. Powers' were first circulated in the late 1960's and Meyer's in 1974. We would like to express our appreciate to both for allowing us to publish their papers and to give them the wider circulation they deserve.

Bob Meyer has pointed out that a more descriptive title for his paper is:

> A GENERAL GENTZEN SYSTEM FOR IMPLICATIONAL CALCULI, PURSUED CURRYESQUELY AND POWERSFULLY, WITH A HINT (BUT NO MORE) FROM BELNAP, WITH APPLICATION TO THE P-W PROBLEM AMIDST RANDOM DWYERIAN (AND OTHER) THOUGHTS, WHICH ENDS IN THE MIDDLE.

While agreeing, as always, with Bob, we have suppressed the longer title to spare bibliographers (such as Wolf) from having constantly to retype the longer title.

Representation of the Fuzziness of the Concept of Relevant Implication within a Multi-valued Model Theory

Introduction

In the first part of this paper (Almog 1976) I suggested that the concept of relevance, as introduced by Anderson and Belnap and developed by Routley, Meyer, Dunn and Urquhart, is a fuzzy concept (in the sense of Zadeh 1975 and Lakoff 1974 .) I claimed that implications are not just either absolutely relevant or absolutely irrelevant. Rather they may be relevant to a degree. Thus to claim that an implication $\ulcorner A \supset B \urcorner$ is either absolutely relevant or absolutely irrelevant seems quite uninuitive, while to claim that there are degrees of relevance of premises to conclusions seems much more plausible.

In the first part I gave some informal arguments supporting this suggestion. In this part I shall try to formalize my arguments and to give to the fuzzy concept relevance a formal treatment within a multi-valued model theory. By using model-theoretic means I can give a precise meta-mathematical account of the suggested fuzziness of the concept of relevant implication.

But apart from a gain in precision and elegance, there are other advantages in a model-theoretic treatment. It helps me to show that a valuation of the sentence "The implication $A \supset B$ is relevant" is a context dependent valuation. This point is very important. I think that the decision whether a premise A is relevant to the conclusion B must be made relative to a point of reference (in the sense of Scott) and so to weaken the general Routley-Meyer assertion "$A \supset B$ is relevant" into "$A \neg B$ is relevant at a point i". This context dependency of the decision as to whether A is

relevant to B has many implications. We mention it because there seems to be a close connection between the fuzziness of relevance and the context dependency of the decision as to whether an implication is relevant. (In connecting fuzziness and context dependency, we follow a suggestion made in Kamp 1975.) It may be that the cause of the fuzziness of the predicate 'relevant' (when assigned to sentences with the form of an implication) is the context dependency of the decision as to whether an implication satisfies the predicate.

Two other implications of a model-theoretic treatment are mentioned but not developed. The first is the similarity between the relativization which the Tarskian concept of truth passed through (from 'truth' to 'truth in a model') and the relativization of the concept of relevance that I suggest (from 'relevance' to 'relevance in a model or at a point of reference'). This parallelism suggests that there is quite an interesting relation between relevance and truth and that maybe one could find interesting results if he were to investigate how to embed a concept of 'relevance' in a theory of 'truth'. (For example, such an investigation has interesting results when the theory of truth is Kripke's (cf. Kripke 1975), but I leave this subject for a future work.)

Finally it might be that the alleged indexicality of the concept of 'relevant implication' will help to connect the logical notion of relevance with the pragmatical-Gricean notion of relevance. The formulation of the fuzziness of relevance within an intensional logic whose first-order reduction in Montague pragmatics and which is context-sensitive may help to connect the two notions.

1. Formal Representation of the Fuzziness of the Concept of Relevance

Given a language $\underset{\sim}{L}$ (to be specified below), we are interested in the sentences having the form "IF A THEN B". This sentence will be formalized in the standard way, i.e., $\ulcorner A \supset B \urcorner$. $\ulcorner A \supset B \urcorner$ is a wff in $\underset{\sim}{L}$ by recursive definitions in the syntax of $\underset{\sim}{L}$ which function on A, B, and $\supset$. Now what interests us is whether "IF A THEN B" is a relevant implication. Defining a connective '$\rightarrow$' let it stand, as Routley and Meyer put it, for relevant implication. Now we would like to have a semantics for '$\rightarrow$'.

The semantics for relevant logic is quite a controversial one (cf. the objections in Scott 1974). In any case we shall use the suggestion of Routley and Meyer. Their informal argument is that in order that "IF THEN B" be relevant, A and B must share a component in a valid conditional. (This is the relevance principle discussed in Meyer 1973 and attributed to Anderson and Belnap 1962.) Formally when a, b, c stand for set-ups (the 'worlds' of Kripke) and R for a ternary accessibility relation, the truth-conditions are '$\rightarrow$' are:

$(A \rightarrow B, a) = 1$ iff for all b, c such that Rabc, $(A, b) = 1 \supset (B, c) = 1$.

Now instead of saying that $(A \rightarrow B, a) = 1$, we can say "THAT A $\supset$ B IS AT a RELEVANT, IS TRUE" or formally if we define Q to be a predicate which has as an extension the set of all relevant implications at a, then

$(Q(p)), a) = 1$ (when $p = A \supset B$).

Thus instead of having a new connective we just check whether an implication satisfies the predicate 'relevant implication' in a certain model (or point of reference). This approach enables us to treat the fuzziness of the concept of relevant implication within a sophisticated multi-valued model theory suggested in Kamp 1975. The basic problem, i.e., that

implications are relevant to a degree, is formulated in the following way: the predicate Q assigned to sentences having the form ⌜A ⊃ B⌝ is a fuzzy predicate. Thus saying that Q(p) is true at a to the degree n is equivalent to the formulation $(A \rightarrow B, a) = n$.

Now let us review Kamp's model theory in which we shall represent the the fuzziness of the concept of relevant implication. The language $\underset{\sim}{L}$ for predicate logic with $\neg$, $\wedge$, $\exists$ and the non-logical symbols which are n-place predicate letters Q_i^n ($n = 1, 2, 3, \ldots$; $i = 1, 2, 3, \ldots$). Now a classical model for $\underset{\sim}{L}$ is a pair (U, F) where: (1) U is a non-empty set, and (2) F assigns to each Q_i^n an n-place relation on U. The satisfaction value of a formula φ of $\underset{\sim}{L}$ in M = (U, F) by an assignment a of elements of U (i.e., $(\varphi)_{M,a}$) is defined:

1. $(Q_i^n(v_{i_1} \ldots v_{i_n}))_{M,a} = 1$ iff $(a(v_{i_1}), \ldots, a(v_{i_n})) \in F(Q_i^n)$
   ($v_1 \ldots v_i$ = variables)
2. $(\neg\varphi)_{M,a} = 1$ iff $(\varphi) = 0$
3. $(\varphi \wedge \psi)_{M,a} = 1$ iff $(\varphi)_{M,a} = 1$ and $(\psi)_{M,a} = 1$
4. $((\exists v_i)\varphi)_{M,a} = 1$ iff for some $u \in U$, $(\varphi)_{M,(a)^u_{v_i}} = 1$.

A sentence φ of $\underset{\sim}{L}$ in M if $(\varphi)_{M,a} = 1$ for some a. Now for any model M, TR_M is the set of true sentences in M and FA_M is the set of the false ones. A partial model for $\underset{\sim}{L}$ is a pair (U, F) where U is a non-empty set and F assigns to each Q_i^n and ordered pair $(F^+(Q_i^n), F^-(Q_i^n))$ of disjoint n-place relations on U. Now satisfaction values are defined:

1. $(Q_i^n(v_{j_1} \ldots v_{j_n}))_{M,a} = 1$ if $(a(v_{j_1}) \ldots a(v_{j_n})) \in F^+(Q_i^n)$
   $(Q_i^n(v_{j_1} \ldots v_{j_n}))_{M,a} = 1$ if $(a(v_{j_1}) \ldots a(v_{j_n})) \in F^-(Q_i^n)$
2. $(\neg\varphi)_{M,a} = 1$ if $(\varphi)_{M,a} = 0$

   $(\neg\varphi)_{M,a} = 0$ if $(\varphi)_{M,a} = 1$.

The same goes for (3) and (4). But now it is clearly possible that certain sentences at M are neither true nor false, so that $TR_M \cup FA_M$ doesn't have the same cardinality as the set of all sentences of $\underset{\sim}{L}$. Now, a partial model M = (U, F) is said to be at least as vague as the partial model M' = (U', F'), i.e., $M \subseteq M'$, if for each Q_i^n:

$$F^+(Q_i^n) \subseteq F'^+(Q_i^n) \text{ and } F^-(Q_i^n) \subseteq F'^-(Q_i^n).$$

To each classical model in $\underset{\sim}{L}$ corresponds a partial one, where for each Q_i^n, $F'^+(Q_i^n) = F(Q_i^n)$ and $F'^-(Q_i^n) = U - F(Q_i^n)$. ((U, F) = classical, (U, F') = partial)

A classical model M is a completion of a partial model M' if M' is at least as vague as (the partial model corresponding to) M.

So far we have presented the basic multi-valued model theory of Kamp 1975 (pp. 134-135) in which the cardinality of the set of truth-values is ≥ 2. But now in order to assign truth-values to formulas containing fuzzy predicates, we need to consider an extension of Kamp's basic model. This is a vague model which consists of a quadruple (M, $\underset{\sim}{S}$, $\underset{\sim}{F}$, pr) where: M is a partial model for $\underset{\sim}{L}$; $\underset{\sim}{S}$ is a set of classical models for $\underset{\sim}{L}$ which are completions of M; $\underset{\sim}{F}$ is a field of subsets over $\underset{\sim}{S}$; for each $\varphi \in \underset{\sim}{L}$ and assignment in the universe of M, $\{M' \in \underset{\sim}{S}: (\varphi)_{M',a} = 1\} \in \underset{\sim}{F}$; pr is a probability measure on $\underset{\sim}{F}$. (For restrictions on 'pr', cf. footnote 2, Kamp 1975, p. 136.)

Now if H is a vague model for $\underset{\sim}{L}$, the degree of satisfaction of φ by a in H, $(\varphi)_{H,a}$ is $pr(\{M' \in \underset{\sim}{S}: (\varphi)_{M',a} = 1\})$.

Vague models can serve us in representing the fuzziness of relevance. But they lack one property and let me explain why it is important. Kamp's basic motivation was to treat within model theory comparative constructions involving vague predicates. Now vague models can represent comparatives as

long as long as when we say "x is more Q than y", we assert that x is in the positive extension of Q (i.e., $F^+(Q)$) and y is in the negative extension. But if both are in the positive extension, vague models cannot represent the comparative. This becomes critical if we want our model theory to represent the sentence "Implication A ⊃ B is more relevant than A ⊃ C". We think that comparison between degrees of relevance is quite a plausible idea and it sounds completely acceptable to compare the relevance of premises if we hold that they are relevant to a degree. But for the moment we shall use vague models in order to represent the fuzziness of the concept of relevance and we shall also give a general account of what should be the truth-conditions of a comparative like "A ⊃ B is more relevant thatn A ⊃ C". (We shall leave aside constructions in which the two compared arguments are in the positive extension of the predicate relevant implication.)

Let us use the letters p, q, r... for sentences of the form A ⊃ B (material implication) and the predicate Q for 'relevant implication'. But in order to represent the fuzziness of relevance (as a concept), we must assume that apart from (1) and (2), (3) is also a possible situation:

(1) $(Q(p))_{M,a} \in F^+(TR_M)$

(2) $(Q(p))_{M,a} \in F^-(TR_M)$; i.e., $(Q(p))_{M,a} \in F^+(FA_M)$

(3) $(Q(p))_{M,a} \in U - F^+(TR_M) \cup F^+(FA_M)$.

It entails that: (4) is a possible situation:

(4) $p \in U - F^+(Q) \cup F^-(Q)$.

(4) says that p is neither absolutely relevant nor absolutely irrelevant. From this observation, we can continue to (5) - (7):

(5) $p \in F^+(Q)$ iff $pr(Q(p))_{M,a} = 1$

(6) $p \in F^-(Q)$ iff $pr(Q(p))_{M,a} = 0$

(7) $p \in U - F^{+}(Q) \cup F^{-}(Q)$ iff $pr(Q(p))_{M,a} = n \qquad 0 < n < 1.$

(7) says that "p is relevant" ($p = A \supset B$) is true to a degree and thus neither in the extension of relevant nor in that of irrelevant.

How does one interpret the probability measure? Following Kamp, we suggest the following approach: the function assigns to each sentence a set of possible worlds. The larger the set, the higher the degree of truth of the sentence. (Kamp attributes this interpretation of probability to Kolmogorov.) Now it is clear that the degree of relevance of a sentence is a function of some factor, i.e., $pr(Q(p))_{M,a} = f(x)$. In the case where the predicate is heavy, we have a strictly monotonic function from the set of all positive real numbers into the interval (0,1). We have positive real numbers because they represent weight. Then for a person u, $pr(H(u))_{M,a} = f(k)$ where k stands for weight.

When the predicate Q is considered, we have another factor. Following Meyer and Routley, it is that the premise and the conclusion in the conditional p must share a component. Formally it means that the world in which the premise (antecedent) holds and the world in which the conditional holds are compossible relative to the world in which the consequent holds. This is the ternary accessibility relation. (Actually other interpretations for 'Rabc' exist, such as 'consistency with', 'intensional conjunction' and 'operation correlating algebraic structures; cf. Routley and Meyer 1973, p. 201.)

Now in order to interpret the Routley-Meyer thesis within a conception of fuzziness, we need to fuzzify the accessibility relation and to have degrees of accessibility. To my knowledge, the first to suggest a fuzzy accessibility relation was Schotch 1975. His suggestion is with regard to

a binary Kripkean relation, but we can use it in a ternary case. Set-ups (worlds) a, b are compossible with respect to c to the degree j iff when A holds in a, B in b and C - which says that A and B are consistent - holds in c and c is accessible to the degree j. This is just an extension of a fuzzification of a binary Kripkean relation which is in fuzzy modal logic: $(\Diamond A, a) = j$ iff there is at least one accessible world to the degree j in which A holds.

Thus we can conclude that the degree of relevance of A to B will be a function of the degree of compossibility just mentioned. Let d stand for degree of compossibility and then: $pr(Q(p))_{M,a} = f(d)$. f will not be a function from positive real numbers. Rather it will be a function from the interval (0,1)(for degrees of compossibility) into itself (to degrees of truth of the sentence). Now it is quite obvious how the comparative involving the predicate Q will be formulated: p is more Q than q iff $(p, q) \in F^{+}(\geq Q)$ iff $pr(Q(p))_{M,a_1} \geq pr(Q(p))_{M,a_2}$. $(a_1 = p; a_2 = q)$

But we can have more than just a comparative which says that it is absolutely true that an implication is more relevant than another. Sometimes we feel that even this assertion is true to a degree and that the alleged difference in degrees of relevance is true to a degree. Then: p is more Q than q is true to a degree iff - formalizing 'p is more Q than q by s - we say:

$$s \in U - F^{+}(\geq Q) \cup F^{-}(\geq Q).$$

This seems to us an approximation of what should be a model-theoretic treatment of degrees of relevance, i.e., of the fuzziness of the predicate 'relevant implication' in the construction "A ⊃ B is a relevant implication".

2. Reflections on the Treatment Suggested.

So far we have assumed that Q is, to use Kampian terminology, a one-dimensional predicate; i.e., when such a predicate is assigned to an argument, the value of the formula is a function of one factor, or alternatively, the argument is considered in the extension of the predicate if it satisfies one requirement. Such predicates are tall (function of height), heavy (of weight), thick, long, fast, etc., in which the membership in the extension is a question of one and only one factor (usually measured numerically).

Now it is not at all clear whether relevant implication (= Q) is or is not in this category. If one assumes that it is not, i.e., that it is a multi-dimensional predicate, then one has to use other models than vague models. One has to use context dependent models because if membership in the set of relevant implications is tested with regard to a set of criteria whose cardinality is greater than 1, then in each context, the criterion might differ. If one considers Kamp's example - clever -, then in context C, the criterion may be good at solving mathematical problems and in C' - $C \neq C'$ - an interesting person.

In Almog 1976 I had assumed that relevant implication is a multi-dimensional predicate. This assumption was made implicitly; I suggested that sentences of the form "$A \supset B$ is a relevant implication" can be modified by hedges such as technically speaking, strictly speaking, etc. Sentences with hedges can get truth values if we presuppose that there is a set of criteria for membership in the predicate's extension (such as primary criteria, secondary criteria, technical criteria).

Now I am not so sure about my suggestion. It seems to me that Q is a

one-dimensional predicate with one criteria, which is the relevance principle of Anderson and Belnap. Whether Q is or is not one-dimensional seems to me a quite trivial problem. In either case it is a predicate which turns a formula, in which it occurs in a predicative position, into a formula with a context dependent truth-value.

Let me develop this point which is very important. I claim that a formula (sentence) of the form "$A \supset B$ is a relevant implication" has a truth-value which is context dependent; i.e., the truth value of $\ulcorner Q(p) \urcorner$ in a world i varies with the contexts i. (We assume the Montague notion of contexts of use, so that in a world i there may be several contexts.) The claim is that one cannot just say "$A \supset B$ is a relevant implication in a world i". Rather one has to say "$A \supset B$ is a relevant implication in a world i in a context j, $j \leq i$". Alternatively if we use the Montague-Scott notion of point of reference (an ordered pair of a world and context of use which specifies the speaker, hearer, place, time and other contextual features) then we can say "$A \supset B$ is a relevant implication at a point k".

Routley and Meyer assumed that "$A \supset B$ is a relevant implication" (or equivalently $A \rightarrow B$) gets a valuation relative to a world, but not to a context. Given a world i and all the contexts of use in i (call it the set J), j^i_k means that j is the k^{th} context in a world i. Then Routley and Meyer assumed that the value of $A \rightarrow B$ at i is constant for all members of J. But I think that relevance of a premise to a consequent (or better, an antecedent to a consequent) is a context dependent matter. The hearer, speaker, place, time and other contextual coordinates have an influence on the decision as to whether the antecedent A is relevant to the consequent B.

One of the basic principles of the classical relevant logic of Anderson

and Belnap 1962 - the entailment theorem - is compatible with our observations. Their theorem says that in order that A → B (or "A ⊃ B is relevant") be true, A must be used in the deduction of B. It seems quite plausible that - given A, B and a possible world i and two contexts of use (in i) j_1^i, j_2^i such that $j_1^i \neq j_2^i$ - it may be that A is used in the deduction of B in one context but not in the other. This is certainly true for indexical entailments as discussed in Montague 1974, p. 230. It is clear that in some cases (contexts) in i "I am hungry" will be used in the deduction of "Thou art hungry". But in some contexts in i in which "thou" referes to another person than the utterer of "I" (in "I am hungry"), the antecedent in the implication "If I am hungry, then thou art hungry" is not relevant to the consequent. Actually there are many combinations in a world i in which "I am hungry" is not relevant to "Thou art hungry" and respectively, combinations in i in which it is relevant. So we see here that relevance is really dependent on contextual features.

But even in conditionals which contain no (explicit) indexicals the same phenomenon occurs. Take David Lewis' counterfactual: "If there were a decision procedure for logic, the war would be over by now". Lewis says the antecedent is irrelevant to the conclusion and one can verify this if he tests this sentence by using either the relevance principle or the entailment theorem (cf. Meyer 1973 p. 169 for a short discussion). But Lewis' categorical statements seem to me to be too categorical. In some contexts one can find a connnection between the antecedent and the consequent. True, these contexts are quite rare, but logic doesn't deal with frequency but with possibilities and it is possible for the Lewis conditional (counterfactual) to be relevant.

I admit that the problem of the relevance of counterfactuals differes from that of material implications, but I have examined this case in order to show that, in some conditionals, relevance is a context dependent matter. This observation though quite speculative at present leads us to consider a possible relativization of the use of Q without context specification. This means that perhaps we should have the notion relevance at a point of reference. It may be found that this is a quite fruitful notion. For example, we might be able to formulate degrees of relevance in this way. Given the set K of points of reference, the higher the probability that a conditional will be relevant at more points of reference, the more we are justified in asserting - as Routley and Meyer did - without context specification "the conditional is relevant". If the probability that a conditional p will be relevant in each k in K (k is a point) is n, then asserting that "p is relevant" will be an assertion true to the degree n.

What is interesting is that this relativization is similar to the Tarskian relativization of truth to truth-in-a-model. Another relativization similar in its nature was made by Kamp, Lewis and Vlach in their departure from a one-dimensional modal logic by saying that instead of asserting that "sentence ⌜A⌝ expresses the proposition A'", we must say "setence ⌜A⌝ expresses the proposition A' in a certain world".

I have a basic intuition connecting all of these relativization, but they (my intuitions) are still not fully developed. The basic idea is to regard the Tarski relativization as the essential (basic) one and to regard all the others as sub-relativizations derived from the one made by Tarksi. In any case this is not the subject of our paper; but one aspect of this

speculation is connected to the present paper. It seems that the need for the idea relevance at a point of reference is a result of truth at a point of reference. This can be made explicit within the theory of truth in Kripke 1975. Just as Kripke tries to connect the predicates true and meaningful, one can try to connect relevant and true (when these predicates describe implications). Let us ask ourselves why the antecedent and the consequent in Lewis' counterfactual are such that the conditional is not a relevant one. Oversimplifying, it seems that there is a certain intuitive notion of field or subject with which the sentences deal such that the antecedent doesn't belong to the field of the consequent. (I am aware of the need to give precise definitions to the terms employed, but this section is only an informal one, without pretending to suggest, at the present, anything more than a speculation.)

Now, here comes Kripke's theory of truth. To use some advice in Kamp 1975 (p. 149, last sentences), I think that in order to see the connection of 'relevant' and 'true', one doesn't have to check only the sentences which are supposed to be true, or relevant, but rather to check the mechanisim or apparatus in which they will be true or relevant. Thus one has to try to see what kind of models we shall have when it would be said that the conditional p is satisfied in a model, but not relevant in that model. Further I think that we ought to check what models in which the antecedent and the consequent are said to be connected would look like.

Given (1) and (2) below, one can find differences in the models which will satisfy these formulas and thus degrees of relevance will be shown to have impact on truth conditions via the structure of the models in which the formulas are satisfied:

(1) If there were a decision procedure for logic, the war would be over by now.

(2) If there were a decision procedure for logic, then Mr. X, who is a logician, would be much more calm by now.

The idea that I have is that Kripkean model theory employed in his truth theory can characterize the differences between models which satisfy true and relevant formulas versus true but non-relevant formulas. (I hope to report on this connection between relevance logic and truth theory in the near future.)

Another advantage of the use of the notion relevant at a point of reference is that it connects formal pragmatics with logic. The logical notion of relevance was, so far, completely without connection to the Gricean notion. But lately attempts have been made to formalize the (pragmatic) notion of relevance. Lakoff and Gordon 1971 provided a first formalization. Later Lakoff 1974a claimed that Grice's notion can be embedded directly in a model theory of natural logic by a use of meaning postulates. At the moment, Grice's notion is still informal. But recently the idea of the topic of a sentence was treated formally and the notion of the topic of a sentence seems to be closely connected with my informal notion that the concept of relevance should be formulated in terms of fields or subjects that sentences deal with. It seems that if Grice's notion of a non-deviation from a topic of a sentence (in order to be relevant) would be appropriately formalized so that his conversational postulates would have a strong basis of meaning postulates, then the logical notion of Anderson and his followers would find some points in common with the pragmatic notion. I have no suggestion in this area, only an intuition of a possible connection.

Appendix

In Almog 1976, appendix 2, I tried to suggest a possible empirical motivation for the fuzziness of the concept of relevance. I used an example from quantum logic. I would like here to develop that idea in light of this paper. Two examples in the literature give additional support to this empirical motivation.

One is Suppes' work on quantum logic in Suppes 1959 and Suppes 1966. In Suppes 1959, he suggested that certain sentences in the natural sciences are neither true no false. He gave as an examples sentences of the form:

$\ulcorner m(a) = 5 \urcorner$ when $\ulcorner m \urcorner$ stands for the mass of the object, $\ulcorner a \urcorner$ as a name of an object.

Suppes claimed that since no unit of measurement is assigned to 5, there is no point in saying that the sentence is true or that it is false. He claimed that such sentences belong to a third class; that of meaningless sentences. Now whether 'meaningless' should be the interpretation of the third value is not the question here. The important fact is that Suppes distinguished a class of neither-true-nor-false sentences. If such a sentence occurs in a conditional, the question as to whether it is relevant to a consequent is not the classical one. If by 5, we meant $\ulcorner 5F\text{'}s \urcorner$, then it may be that the sentence is relevant to the consequent. But if in saying 5, we meant $\ulcorner 5G\text{'}s \urcorner$, maybe G is such a predicate as to make the conditional irrelevant. (Would "If the mass of the object is 5 seconds, then the boat will sink down" be a relevant implication? I think that since the antecedent is deviant, the answer is a vague one.

Suppes 1966 deals with the non-classical character of quantum logic. Suppes observes that the joint probability of two events doesn't necessarily

exist in quantum mechanics. Suppes shows that one can derive from this the fact that the functional logic of quantum logic is not a Boolean algebra. As far as we are concerned, we need only to consider the case in which one is to determine the joint probability of a particle being at t in a certain region and having a momentum lying in a certain interval at the same time t. Since one cannot do that by standard computation, the situation is not classical. Then if the two events mentioned are formalized by A and by B and the conjunction A&B is an antecedent in a conditional, since pr(A&B) is not determined necessarily, the question as to whether this antecedent is relevant to a given consequent is not always answerable or, at least, the answer is not necessarily either 'absolutely relevant' or 'absolutely irrelevant'.

My last point concerning the suggestion that the quantum conditional is an <u>empirical</u> example of degrees of <u>relevance</u> is that we saw that Lewis counterfactuals may be relevant to a degree. Given his antecedent "if there were a decision procedure to logic", different consequents (combined with this antecedent) will find the antecedent relevant to different degrees. Now since Hardegree 1975 has shown that the quantum conditional (the Sasaki hook) is a counterfactual (Stalnaker) conditional, we may derive the conclusion that in the quantum conditional, just as in other counterfactuals, there may be degrees of relevance of an antecedent to a consequent.

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A Simple Defense of Material Implication

If any truth-values are consistently assigned to a conditional of the form 'If p then q' (for which modus ponens and modus tollens are valid argument-forms and affirming the consequent is a fallacy), then 'If p then q' and 'p ⊃ q' may be shown to be logically equivalent. The argument is straightforward, but before proceeding further it would be well to state the assumptions explicitly.

Ordinarily, an argument is said to be invalid if and only if the form of that argument has at least one substitution instance with true premisses and a false conclusion. Also, logicians agree that 'If p then q' is false when p is true and q is false. As both of these conditions are not controversial, they will be presupposed without comment. Two remaining assumptions seem initially plausible, and, for the sake of this defense, will be assumed as well.

First, a valid form of ordinary language argumentation has at least one substitution instance with true premisses and a true conclusion. A good argument for such a necessary condition for validity has been given by George A. Clark: "A valid argument establishes the truth of its conclusion if its premisses are true; it cannot establish its conclusion on the basis of premisses asserted to be false or on the basis of premisses asserted to be true and false."[1] Let this be called a necessary condition for validity.

Second, 'If p then q' is assumed to have its truth-values consistently but arbitrarily assigned; in other words, 'If p then q' is assumed to be truth-functional without prejudice toward the assignment of a particular truth value for the whole expression. For convenience, let the value '1'

be assigned in case both the antecedent and consequent are true, the value 'F' in case the antecedent is true and the consequent is false (according to an assumption given above), the value '3' in case the antecedent is false and the consequent is true, and the value '4' in case both antecedent and consequent are false. In this manner, the values '1', '3', and '4' taken separately are not presupposed to be true or presupposed to be false, and 'If p then q' seems to be initially only partially truth-functional. From the above conditions, this defense of material implication will take three short steps.

Step 1. An ordinary language argument-form in terms of modus ponens is arrayed in Table 1.

TABLE 1

	Guide Columns		First Premiss	Second Premiss	Conclusion
	p	q	If p then q	p	q
1.	T	T	1	T	T
2.	T	F	F	T	F
3.	F	T	3	F	T
4.	F	F	4	F	F

Lines 2, 3, and 4 of Table 1 all have a false premiss. By Clark's necessary condition for validity, line 1 then must have true premisses. Therefore, the value '1' is 'true'.

Step 2. Modus tollens as an ordinary language argument form is given in Table 2.

TABLE 2

	Guide Columns		First Premiss	Second Premiss	Conclusion
	p	q	If p then q	not-q	not-p
1.	T	T	1	F	F
2.	T	F	F	T	F
3.	F	T	3	F	T
4.	F	F	4	T	T

Lines 1, 2, and 3 of Table 2 all have a false premiss. Again, by the necessary condition for validity line 4 must have true premisses; thus, '4' answers to 'true'.

Step 3. An ordinary language argument-form for the fallacy of affirming the consequent is displayed in Table 3.

TABLE 3

	Guide Columns		First Premiss	Second Premiss	Conclusion
	p	q	If p then q	q	p
1.	T	T	1	T	T
2.	T	F	F	F	T
3.	F	T	3	T	F
4.	F	F	4	F	F

Since this argument-form is invalid, at least one substitution instance has true premisses and a false conclusion. Line 3 of Table 3 meets this requirement if and only if the value of '3' is taken as 'true'.

Clearly then on our assumptions the derived truth values for 'If p then q' are identical to 'p ⊃ q'. The resultant truth-values (when considered in the order of Step 1, the sufficiency condition for the falsity of 'If p then q', Step 3, and Step 2) correspond exactly to the four respective lines of the truth-table for 'p ⊃ q'.

This simple defense of material implication helps to clarify the debate between the orthodox logicians, who claim that 'If p then q' and 'p ⊃ q' are not interderivable, and the nonorthodox logicians, who claim that the two expressions are interderivable. In order to maintain the orthodox position, 'If p then q' cannot be assumed to be truth-functional without prejudice toward the truth-values of the various substitution instances, for if modus ponens and modus tollens are valid forms and affirming the consequent is invalid, then conditionals in these arguments are wholly truth-functional cases of material implication. Consequently, the orthodox logician must deny that this class of conditionals is not truth-functional in any consistent sense on three lines of the truth-table.

NOTES

[1]George A. Clark, "Note on False Premises and True Conclusions," Journal of Philosophy, LV, 26 (Dec. 18, 1958), p. 1148. If, however, one objects to this stringent condition, it can be dropped in favor of less rigorous conditions without impugning the validity of the rest of the argument. For example, W. S. Thomblison has pointed out that instead of the stated necessary condition for validity, one need assume only that modus ponens and modus tollens have sound instances.

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A Three-Valued Interpretation for a Relevance Logic

In this paper an entailment relation which holds between certain propositions of the propositional calculus will be defined both syntactically and semantically. Some theorems about this relation will show why one could not follow Lewis to prove that a contradiction entails, for the notion of entailment discussed below, every proposition.

We will develop a system RC in which the primitive symbols are the five symbols

$$\vee \quad \cdot \quad - \quad (\quad)$$

and the propositional variables

$$p_1 \quad p_2 \quad p_3 \quad \cdots \quad .$$

The formation rules of RC are:

1) A variable standing alone is a well-formed formula (wff).

2) If A and B are well-formed (wf) then (A v B) is wf.

3) If A and B are wf then (A · B) is wf.

4) If A is wf then -A is wf.

5) If A is wf it is so in virtue of 1) - 4).

We will let capital letters with or without subscripts be variables which range over occurrences of wffs. Following Church (see Introduction to Mathematical Logic, pp. 135-6) we will say that X is the full disjunctive normal form of A (the FDNF of A), where $p_{i_1}, \ldots, p_{i_n}$ are all and the only

propositional variable constituents of A, and $i_j < i_k$ if $j < k$, if and only if i) X is a disjunction of at most 2^n conjuncts such that each conjunction is identical to $(C_{m_1} \cdot (C_{m_2} \ldots C_{m_n}) \ldots)$ $(1 \leq m \leq 2^n)$, where C_{m_j} $(1 \leq j \leq n)$ is either p_{i_j} of $-p_{i_j}$ and ii) '((A · X) v (-A · -X))' is a tautology. By the full disjunctive normal form of A relative to B (the FDNF of A/B) we will mean the FDNF of A if every p_i in B is a p_i in A. If $p_{i_1}, \ldots, p_{i_n}$ are all and the only propositional variables which occur in B but not in A then by the FDNF of A/B we will mean the FDNF of

$$(A \cdot ((p_{i_1} \text{ v } -p_{i_1}) \cdot ((p_{i_2} \text{ v } -p_{i_2}) \ldots (p_{i_n} \text{ v } -p_{i_n}) \ldots)$$

Let us now define a relation which we will call syntactic-relevance-entailment and denote by ' $\rightarrow$ '. $A \rightarrow B$ if and only if

i) Every disjunct in the FDNF of A/B is a disjunct in the FDNF of B/A;

ii) There is at least one disjunct in the FDNF of A/B; and

iii) Every p_i in B is a p_i in A.

To define semantic-relevance-entailment, denoted by ' $\Rightarrow$ ', we will use the notion of a valuation of a wff of RC. Let V be a valuation of A if V is a function which i) assigns 0, 1 or 2 to each p_i in A, ii) assigns the same value to different occurrences, if any, of the same p_i in A and iii) assigns 0, 1 or 2 to A as directed by the following tables:

v	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

·	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

	-
0	1
1	0
2	2

$A \Rightarrow B$ if and only if

i) Every valuation that assigns 0 to A assigns 0 to B; and

ii) There is at least one valuation which assigns 0 to A.

The above notions of syntactic-relevance-entailment and semantic-relevance-entailment are extensionally equivalent. To show this (Theorem 1, below) we will make use of the following lemmas.

Lemma 1. If for every p_i in X $V(p_i) = 0$ or 1 then $V(X) = 0$ or 1. Proof: By strong induction on the number of symbols in X.

Lemma 2. If for every p_i in X ($V(p_i) = 0$ or 1, then $V(X) = 0$ if and only if V(the FDNF of X) = 0 and $V(X) = 1$ if and only if V(the FDNF of X) = 1. Proof: Standard result.

Lemma 3. If $V(X) = 0$ or 1 and p_i is a wf part of X then $V(p_i) = 0$ or 1. Proof: By using strong induction on the number of symbols in X we will prove that if p_i is a wf part of X and $V(p_i) = 2$ then $V(X) = 2$. i) Suppose that there is one symbol in X. Then $X = p_i$. If $V(p_i) = 2$ then $V(X) = 2$. ii) By the induction hypothesis if there are m symbols in X, where $m < n$, then if p_i is a wf part of X and $V(p_i) = 2$ then $V(X) = 2$. Consider a formula Y in which there are n symbols where $m < n$. We need to show that if there is a p_i in Y such that $V(p_i) = 2$ then $V(Y) = 2$. There are three cases to consider. a) $Y = (Y_1 \vee Y_2)$. Suppose there is a p_i in Y it must be either Y_1 or Y_2. If there is a p_i in Y_1 such that $V(p_i) = 2$ then by the induction hypothesis $V(Y_1) = 2$. But if $V(Y_1) = 2$ then $V(Y) = 2$. By similar reasoning if there is a p_i in Y_2 such that $V(p_i) = 2$ then $V(Y) = 2$. b) $Y = (Y_1 \cdot Y_2)$. Similar to case a). c) $Y = -Y_1$. Similar to case a).

Lemma 4. If V(the FDNF of A/B) = 0 then V(A) = 0. Proof: i) Suppose that every p_i in B occurs in A. Then the FDNF of A/B is identical to the FDNF of A. But if V(the FDNF of A) = 0 then by lemmas 2 and 3 V(A) = 0. ii) Suppose

that $p_{i_1}, \ldots, p_{i_n}$ are all and the only variables that occur in B but not in A. Then the FDNF of A/B is identical to the FDNF of $(A \cdot ((p_{i_1} \vee -p_{i_1}) \ldots (p_{i_n} \vee -p_{i_n}) \ldots)$. By lemmas 2 and 3 if V(the FDNF of A/B) = 0 then $V((A \cdot ((p_{i_1} \vee -p_{i_1}) \ldots (p_{i_n} \vee -p_{i_n}) \ldots) = 0$ and thus (by the tables) V(A) = 0.

Lemma 5. If V(the FDNF of A/B) = 1 then V(A) = 1. Proof: Similar to the proof for Lemma 4. (Note that if $V((A \cdot ((p_{i_1} \vee -p_{i_1}) \ldots (p_{i_n} \vee -p_{i_n}) \ldots) = 1$ then V(A) = 1.)

Lemma 6. If the FDNF of A/B is non-empty then there is a valuation which assigns 0 to A. Proof: If the FDNF of A/B is non-empty let $(C_{1_1} \cdot (C_{1_2} \cdot \ldots \cdot C_{1_n}) \ldots)$ be the left-most disjunct of the FDNF of A/B, where $p_{i_1}, \ldots, p_{i_n}$ are the variables which occur in A or B. If $C_{1_r} = p_{i_r}$ let $V(p_{i_r}) = 0$; if $C_{1_r} = -p_{i_r}$ let $V(p_{i_r}) = 1$. Then $V((C_{1_1} \cdot (C_{1_2} \cdot \ldots \cdot C_{1_n}) \ldots) = 0$. Since $C_{j_r} = p_{i_r}$ or $-p_{i_r}$ $(1 \leq j \leq 2^n; 1 \leq r \leq n)$ V(any disjunct in the FDNF of A/B) = 0 or 1. So V(the FDNF of A/B) = 0. By Lemma 4 V(A) = 0.

Lemma 7. If there is a valuation V such that V(A) = 0 then the FDNF of A/B is non-empty. Proof: Suppose V(A) = 0. Then by lemmas 2 and 3 V(the FDNF of A) = 0. Since there is nothing assigned to the empty symbol by V, the FDNF of A is non-empty. If the FDNF of A is non-empty then the FDNF of $(A \cdot ((p_{i_1} \vee -p_{i_1}) \ldots (p_{i_n} \vee -p_{i_n}) \ldots)$ is non-empty, where p_{i_j} $(1 \leq j \leq n)$ does not occur in A. So the FDNF of A/B is non-empty if V(A) = 0.

Theorem 1. $A \rightarrow B$ if and only if $A \Rightarrow B$. Proof: Consider the three conditions: a) every disjunct in the FDNF of A/B is a disjunct in the FDNF of B/A,

b) there is at least one disjunct in the FDNF of A/B and c) every p_i in B is a p_i in A. We must show that these conditions are met if and only if the following two conditions are met: d) every valuation which assigns 0 to A assigns 0 to B and e) there is a valuation which assigns 0 to A.

i) (If a), b) and c) then d).) Assume that a), b) and c) are true and that V(A) = 0. By lemmas 2 and 3 V(the FDNF of A) = 0. Since c) is true the FDNF of A is identical to the FDNF of A/B. So V(the FDNF of A/B) = 0. So V assigns 0 to at least one of the disjuncts of the FDNF of A/B. By a) V assigns 0 to at least one of the disjuncts of the FDNF of B/A. But then V assigns 0 or 1 to each of the disjuncts of the FDNF of B/A and thus V assigns 0 to the FDNF of B/A. By Lemma 4 if V(the FDNF of B/A) = 0 then V(B) = 0. ii) (If a), b) and c) then e).) Follows from Lemma 6.

iii) (If d) and e) then a).) We will show that if a) is false then d) is false. Suppose there is a disjunct of the FDNF of A/B which is not a disjunct of the FDNF of B/A. Let V assign 0 to this disjunct. Then V assigns 1 to every other disjunct in the FDNF of A/B and 1 to every disjunct in the FDNF of B/A. So V(the FDNF of A/B) = 0 and V(the FDNF of B/A) = 1. By lemmas 4 and 5 V(A) = 0 and V(B) $\neq$ 0. iv) (If d) and e) then b).) Follows from Lemma 7. v) (If d) and e) then c).) We will show that if c) is false then d) is false. Assume that p_i is in B but not in A. Let V(A) = 0 and $V(p_1) = 2$. By Lemma 3 V(B) = 2.

Theorem 2. (Simplification) If there is a valuation which assigns 0 to (A · B) then (A · B) $\Rightarrow$ A. Proof: By examination of tables.

Theorem 3. (Commutation) If there is a valuation which assigns 0 to (A · B) then (A · B) $\Rightarrow$ (B · A). Proof: By examination of tables.

Theorem 4. (Addition) If there is a valuation which assigns 0 to A and if every p_i in B is a p_i in A then $A \Rightarrow A \vee B$. Proof: Assume that V(A) = 0 and that every p_i in B is a p_i in A. By lemmas 1 and 3 V(B) = 0 or 1. So V(A v B) = 0.

Theorem 5. (Adjunction) If $A \Rightarrow B$ and $A \Rightarrow C$ then $A \Rightarrow (B \cdot C)$. Proof: By examination of tables.

Theorem 6. (Disjunctive Syllogism) If there is a valuation which assigns 0 to $(-A \cdot (A \vee B))$ then $(-A \cdot (A \vee B)) \Rightarrow B$. Proof: By examination of tables.

Theorem 7. (Transitivity of Entailment) If $A \Rightarrow B$ and $B \Rightarrow C$ then $A \Rightarrow C$. Proof: i) if $A \Rightarrow B$ then there is a valuation which assigns 0 to A. ii) If V(A) = 0 then if $A \Rightarrow B$ V(B) = 0. But if V(B) = 0 and $B \Rightarrow C$ then V(C) = 0. So if V(A) = 0 and the antecedent of Theorem 7 is true V(C) = 0.

SYMPOSIUM/ The P-W Problem

Now that the four color (or colour) map theorem has finally been proven, mathematicians and logicians need another recalcitrant problem to occupy their spare time. Such a problem should have at least three properties: 1) it should be relatively tirival, in that once it is solved, not much has been learned; 2) it should be easily understandable, so that the layperson can understand what is being investigated (though perhaps (s)he may be unclear as to why anyone bothers); and 3) it should be a difficult problem, so that people expend a great deal of effort. (Perhaps we could add: 4) the effort to solve the problem should bring about conceptual advances in the creation of novel strategies and theories, so that even failure to solve the original problem leaves one better off than before.)

We suggest the P-W problem, now that the admissibility of gammma has been cleared up, as our suggestion as to the approporiate problem for logicians, anyway, to wrack their brains over. The P-W (or $T_{\rightarrow}$-W) puzzle itself is stated in all five of the following papers. Here we would only state that the problem satisfies the three (four?) criteria listed above and that the following papers include all of the results known to us at this time about the problem, except for Dwyer's results, which are listed in Meyer's paper. (Hopefully we will be able to include these results in a future issue.)

One other pertinent fact may be quoted from Entailment, volume one, page 95: "A matrix of Lemmon et al. 1956 shows that no instance of $A \rightarrow A$ in which each variable occurs exactly once in A is provable from the two transitivities alone." The reference is to: E. J. Lemmon, C. A. Meredith, D. Meredith, A. N. Prior and I. Thomas, "Calculi of Pure Strict

Implication" (mimeographed 1956), printed in Philosophical Logic, edited by J. W. Davis, D. J. Hockney and W. K. Wilson, Dordrecht (D. Reidel Publishing Company) 1969, pp. 215-250. The matrix is on page 230.

We might also mention a related conjecture that readers might work on: In Relevant Logics and Their Rivals, forthcoming, Richard Routley has conjectured that the "ideal" set-theoretical extension of T-W is absolutely consistent, where the set theory concerned "differs from that discussed in Fraenkel Bar Hillel 58 only in that the underlying logic is a relevant logic and that the axioms of comprehension and extensionality are formulated with connectives $\rightarrow$ and $\leftrightarrow$ appropriately replacing the classical connectives $\supset$ and $\equiv$."

Given the recent progress in showing the absolute consistency of relevant arithmetics, stemming from Meyer's work, this problem may however already be solved.

The Newsletter would be glad to report any advances on these problems that come to our attention. In the event of a solution, we will have to come up with a replacement puzzle. (In the case of duplicate entries, the prize will be awarded to the entry with the earliest postmark.)

On P-W

Consider the following system, hereafter called system 1-2-3:

system
1. $A \to A$
2. $(A \to B) \to ((B \to C) \to (A \to C))$
3. $(B \to C) \to ((A \to B) \to (A \to C))$
4. Rules: MP, substitution

The original problem (not solved) was to show that no pair of theorems $\alpha \to \beta$, $\beta \to \alpha$ with $\alpha \neq \beta$ could be derived in this system. I have shown that the first axiom may be struck out - that if such a pair can be derived, it can be derived using only axioms 2 and 3. It follows from this result that no such pair can be derived unless some substitution instance of axiom 1 can be derived using only axioms 2 and 3. Thus the problem appears to be reduced to that of showing this latter.

In fact, suppose $\alpha \to \beta$ and $\beta \to \alpha$ are deduced from axioms 2 and 3. Then $(\alpha \to \beta) \to ((\beta \to \alpha) \to (\alpha \to \alpha))$ gives an instance of axiom 1. Thus if no $\alpha \to \alpha$ can be deduced from 2 and 3, neither can any pair $\alpha \to \beta$ and $\beta \to \alpha$.

I shall show that no relevant pair $\langle \alpha \to \beta, \beta \to \beta \rangle$ can be gotten from 1-2-3 that cannot be gotten from 2-3. In the relevant pairs $\alpha \neq \beta$ and so neither member of the pair is an instance of axiom 1. I shall prove that every theorem of system 1-2-3 which is not an instance of axiom 1 is derivable without use of axiom 1, i.e., from 2 and 3 only.

Using variables to range over formulae of our system, I shall write $y = x \circ z$ for $x = z \to y$. In other words, if z is the antecedent of x, then xoz shall be x's consequent. Otherwise xoz shall be undefined.

This operation may be called the modus ponens product or the removal product (the antecedent is removed from the conditional to get the

consequent, when MP is performed). Thus x o z might be read as 'ex remove zee' and represents the result of using a as major premise and z as minor premise in a modus ponens. For instance (a → b) o a = b.

Let X be a class of formulae, and Z be a class of formulae. By XoZ we represent the class of all formulae u such that there are x ∈ X, z ∈ Z such that x o z. If x o z is never defined for x ∈ X, z ∈ Z, then XoZ is empty.

Let A be the class of instances of axiom 1, B that of instances of axiom 2, C that for axiom 3.

Since every theorem of 1-2-3 is either ∈ A, ∈ B, ∈ C or is a MP product of members of A, B and C, a study of the MP products of A, B, C may indicate something about the theorems of 1-2-3.

The result I've claimed will be proven if it is shown that every product of A, B, and C is either contained in A or is contained in a product of B and C alone (or both).

But first we need some preliminary results.

Result 0b) BoA ⊂ A (I use '⊂' for set inclusion, proper or improper)

Pf: Let x ∈ BoA. ∃y, z such that y ∈ B, z ∈ A, x = yoz.

∀ u such that z = (u → u), since z is in A. ∃a,b, c such that y = (a → b) → ((b → c) → (a → c)). But y = z → x. Thus y = ((u → u) → x) = ((a → b) → ((b → c) → (a → c))) and so a = b and x = ((b → c) → (a → c)) = ((a → c) → (a → c)).

Thus x ∈ A. Q. E. D.

Result 1) For any class X, AoX = X

Pf: 1st ⊂. Let x ∈ AoX. Then x = uoz, u ∈ A, z ∈ X. ∴ u = (y → y) = (z → x) ∴ z = y = x. But z ∈ X, so x ∈ X.

2^{nd} $\supset$. Let $x \in X$, Then $(x \to x) \in A$. $x = (x \to x)ox$ and thus $x \in AoX$. Q. E.

Result 2) For any X, $BXA \subset X$. (I associate to the left.)

Pf: Let $x \in BXA$. $x = uovow$, $u \in B$, $v \in X$, $w \in A$.

$\therefore u = v \to (w \to x) = (a \to b) \to ((b \to c) \to (a \to c))$. But $w \in A = u \to u = b \to c$.

$\therefore b = c$. $\therefore (a \to b) = (a \to c) = v = x$. But $v \in X$, so $x \in X$. Q. E. D.

Result 3) $CXY = BYX$

Pf: Let $z = dxy$, $d \in C$, $x \in X$, $y \in Y$.

$d = (b \to c) \to ((a \to b) \to (a \to c))$, for some a, b, c

$x = (b \to c)$

$y = a \to b$ for the same a, b, c

$z = a \to c$

Now let $e = (a \to b) \to ((b \to c) \to (a \to c)) \in B$.

Obviously: $z = eyx$

For $\supset$. Let $z = dyx$, $d \in B$, $x \in X$, $y \in Y$.

Let $d = (a \to b) \to ((b \to c) \to (a \to c))$

Then $y = a \to b)$

$x = b \to c$

and $z = a \to c$

If $e = (b \to c) \to ((a \to b) \to (a \to c)) \in C$

Obviously: $z = exy$. Q. E. D.

Result 4) $BXYZ = Y(XZ)$

Pf: Let $z = [(a \to b) \to ((b \to c) \to (a \to c))]\, o(a \to b)o(b \to c)\, o\, a$.

Then $z = c = (b \to c)o[(a \to b)oa]$

On the other hand:

Let $\alpha = y(xz) = y((z \to u)a) = (u \to \alpha)((z \to u)z)$

Then $\alpha = [(z \to u) \to ((u \to \alpha) \to (z \to \alpha))]o(z \to u)o(u \to \alpha)oz$

which $\in BoXoYoZ$. Q. E. D.

Result 0C) $CoA \subset A$ Pf: cf. 0b Q. E. D.

Result 5) $CXA \subset X$ Pf: cf. 2 Q. E. D.

Result 6) $CXYA = X(YZ)$ Pf: cf. 4 Q. E. D.

The proofs of 0c, 5, 6 are similar to those of 0b, 2, 4.

I here collect the initial results for easy reference.

0b) $BA \subset A$

1) $AX = X$

2) $BXA \subset X$

3) $CXY = BYX$

4) $BXYZ = Y(XZ)$

I note further that $X \subset Y$ implies $XZ \subset YZ$ and $ZX \subset ZY$.

0c) $CA \subset A$

5) $CXA \subset X$

6) $CXYZ = X(YZ)$

By means of these formulae, it may be shown (as I shall show) that, if α is any class expressible as a product of A, B, and C, then α is included in some class which either is A or is expressible as a product of B and C (without A). This means that if x is any theorem of the original system 1-2-3, then x is either an instance of axiom 1, or x amy be proven without using axiom 1.

Example:

B(BB)BBA(B(CA)B) is a class (possibly empty) expressed as a product whose utlimate factors are each A, B, or C. Our theorem (yet to be proven) states this class is included in (and may be identical to) some class which is expressible as A alone or without using A.

In fact result 0c tells us that $CA \subset A$. Consequently the above $\subset$ B(BB)BBA(BAB). Result 0b) is that $BA \subset A$ also. So the above $\subset$ B(BB)BBA(AB).

Since I am associating to the left, the other 'BA' is really '((...B)A)' and so cannot be treated in the same manner. Result 4 is that BXYZ = Y(XZ) and so the last above now $\subset$ B((BB)B)A(AB).

This last step may more prespicuously be represented as:

$[B_1(BB)_2B_3B_4]A(AB) = [B_3((BB)_2B_4)]A(AB)$.

Result 1 is that AX = X and so the last is B((BB)B)A(B).

Result 2 is that BXA $\subset$ X and so our case is $\subset$ ((BB)B)B or, in other words, BBBB. So our original class is included in BBBB and its members may be proven without use of A (axiom 1).

Theorem: Every class which is a product of classes A, B, and C is either included in A or is included in a class which is a product of B and C.

Definition. A, B, and C are "products of A, B, and C" and if X and Y are "products of A, B, and C", then so is XoY a "Product of A, B, and C".

Definition. B and C are "products of B and C" and if X and Y are, so is XoY.

Proof of theorem:

It suffices to show that if any product expression α using 'A' 'B' and 'C' as its only ultimate factors be written down, then if the expression is not simply 'A' and if the expression contains at least one 'A' (token), then a shorter expression β may be found such that $\alpha \subset \beta$ is true and such that β also uses only 'A' 'B' and 'C' as ultimate factors. The word 'shorter' is explained in terms of 'length'.

The length of a product expression is the sum of the number of 'A''s, the number of 'B''s, and the number of 'C''s.

In other words, if α is not 'A' and contains 'A', we replace α by a shorter expression, an expression involving fewer ultimate factors. If we still do not have 'A' or an 'A'-free formula, we replace it by a still shorter.

If such replacement is always possible, our theorem is proven, since a series of decreasing finite lengths all ≥ 1 must have a least member.

Suppose then we are given an expression α. If it is 'A' or contains no 'A', we are through. If it contains at least one 'A' and has length $l \geq 1$, number the A's from left to right.

Case 1) There is an A_j such that there is an A_k such that '$(A_j o A_k)$' is a part of α. If so, replace '$(A_j o A_k)$' by '(A_k)' using result 1.

Case 2) There are no A_j, A_k as in case 1. Consider any A_j. It occurs in an expression $((....)o(A_j))$ or in an expression $((A_j)o(....))$. I am supposing all parentheses restored. Now, in either case '(....)' is not 'A_k' for any k. (since then we would be back to case 1). If we are concerned with at least one j such that $((A_j)o(....))$ occurs, we replace this by (....), using result 1.

Case 3) No case of $((A_j)o(....))$ occurs. For every k, A_k occurs in an expression of form $((....)o(A_k))$, where (....) is not (A_1), for any 1. We define, for any i, the degree of A_i as the number of A's in the expression (....) such that $((....)oA_i)$ is in α. By noting that if A_i's degree is $n \geq 1$, A_j in (....) must have a degree less than n, we see that some A_k must have degree zero. Let us pick such an A_k. Then $((....)oA_k)$ occurs in α and there are no A's in (....) which is thus a product of B's and C's. If (....) is of length 1, it is either B or C. $BA \sqsubset A$ and $CA \sqsubset A$ and so we suppose (....) is of length greater than 1.

Let (....) = $((\gamma)o(\delta))$. If ν is of length 1, then we have either $(B(\delta))A_k = B(\delta)A \sqsubset (\delta)$ by result 2, or $C(\delta)A \sqsubset (\delta)$ by result 5.

Suppose ν of length 2. We have $\gamma = BX$ or $\gamma = CX$. Then $BX\delta A = \delta(XA)$ or $CX\delta A = X(\delta A)$ shortens our formula.

We may suppose ν of length greater than 2. Let $((\nu)o(\delta))oA_k$ be re-written in a Polish notation as ooo.... ....A_k. In other words, if γ' is Polish for γ, and δ' is Polish for δ, we have $oo\gamma'\delta'A_k$. Proceeding from the left end of the formula we find after the big dots an ultimate factor other than A_k, call it D. D is either a B or a C. We have oo...oD...A_k. Before D there is a strong of big dots. There are at least three; one for A_k, one for δ' and one for the rest of ν'. Consequently, choosing the three closest to D we have a formula of the form oooDXYZ = ((DoX)oY)oZ = DXYZ = either X(YZ) or Y(XZ). Q. E. D.

Thus we have proven that:

{theorems of 2-3} $\cup$ {instances of axiom 1} = {theorems of 1-2-3}.

It suffices for a (negative) solution to the original problem to show further that: {theorems of 2-3} $\cap$ {instances of axiom 1} = $\emptyset$. However, I have not succeeded in showing this.

Probably the most efficient approach would be to try matrices. But not being very well acquainted with matrices I have been trying a different approach which has turned out to be pretty inefficient. Nonetheless, I have gotten some partial results and these indicate that, if relevant matrices are not as easy to find as might be hoped, the present methods are not irrelevant.

Suppose some instance of axiom 1 is a theorem of system 2-3, contrary to what we would like to prove. Then some number l is the smalles such that a product expression in B and C is of length l and denotes a class not disjoint from A. To show what we want, it suffices to show that no such l and no such shortest product exists.

What would such a shortest product expression look like? It could

not be of the form BX. For suppose BX not disjoint from A. Then $b_1 o x_1 = a_1$ and $((d \to e) \to ((e \to f) \to (d \to f)))o(d \to e) = a_1 = ((e \to f) \to (d \to f))$. So $d = e$ and $(d \to e) = x_1 \in A$. So if BX isn't disjoint from A, neither is X, and so BX is not the shortest (one of the shortest) of this sort. Similarly for CX.

This suggest we consider cases in which C or B are the right factor, but not the left. Simplest forms of this sort are $C^1 = C$, $C^2 = CoC$, $C^3 = (CoC)oC$, $((CoC)oC)oC = CoCoCoC = C^4$, etc., excluding cases C^1 and C^2; and similarly B^n for $n \geq 3$. Of these two series, C^n turns out to be relatively trivial; B^n turns out to require hard work.

Let us consider C^n first. We know that $CXYZ = X(YZ)$. So $CCCC = C(CC)$. $C^5 = C(CC)C$. $C^6 = C(CC)CC = CC(CC)$. [Note: $C(CC)_1C_2C_3 = (CC)_1(C_2C_3) = CC(CC)$.] Next $C^7 = CC(CC)C = C((CC)C)$. $C^8 = C((CC)C)C$. $C^9 = C((CC)C)_1C_2C_3 = CCC_1(C_2C_3) = C(C_1(C_2C_3))$, where subscripts have been added to aid the comprehension of the steps. $C^{10} = C(C(CC))C$. $C^{11} = C(C(CC))CC = C(CC)(CC)$. $C^{12} = C(CC)(CC)C = CC((CC)C)$. $C^{13} = CC((CC)C)C = C\ulcorner((CC)C)C\urcorner = C\ulcorner CCCC\urcorner = C(C(CC)) = C^9$. So $C^{13} = C^9$.

Consequently it suffices to examine the forms of the wffs in C^3, C^4,... and C^{12}. I will not show this work fully (the work itself is given below without explanation); suffice it that the work is trivial and the result is that C^n is disjoint from A for every n.

Here I show some calculations copied from a notebook.

1) $\dfrac{yz \quad \cancel{xy} \; xz}{\underset{a}{b} \quad \underset{yz}{c}}$ $\qquad\qquad$ $\underset{a}{\underset{23}{\dfrac{23 \quad 12 \quad 13}{a \quad y \quad z}}}$ $\qquad\qquad$ 3) $\dfrac{2 \quad 3}{(x \quad 12)(x \quad 13)}$

2) $\underset{x}{\underset{12}{(x \quad y \quad xz)}}$ $\underset{x}{\underset{13}{}}$

4) bc ab a c
x → 1 (y z)
x → 1 (Uy Uz)
bc ab Ua Uc

5) 23 1213
b c
a b (Ua Uc)
a 23 Ua !!(12 13)

6) yz xy xz
a 2 3
!!a !!(12 13)
yz 1xy1xz

7) bc ab ac
U y z
U (1 x y 1 x z)

8) vw !!v !!w
b c
(1 → x ab)(1 → x ac)

9) yz xy x z
1 (X a(v w))
1 (X a(!!v !!w)
y(vw) xy x( )

10) y → (vw)
xy → (Uv Uw)

11) x 23
x(!!12 !!13)

12) b c
U (1 ab) → U(1 ac)

The reader should skip the above, which are included in the hope that they will be suggestive to anyone who is concerned to work out the result for himself

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So much for C^n.

Now consider $B^n = (((B)oB)....)oB) = \underbrace{BBBBBB...B}_{n\ B's}$. In this case we have BXYZ = Y(XZ). Consider B^{10}. $B^{10} = \underbrace{B\ B's}_{10}$ $= B_1B_2B_3B_4B_5B_6B_7B_8B_9B_{10} = B_3(B_2B_4)B_5B_6...B_{10} = B_5(B_2B_4B_6)B_7B_8B_9B_{10} = B_7(B_2B_4B_6B_8)B_9B_{10} = B_9(B_2B_4B_6B_8B_{10})$. Here all the even B's have been lined up in the second term. We could proceed next to $B^{10} = B_9(B_6(B_4B_8)B_{10}$ by working on the second term (a B^5). However we note the following: $B^{10} = B(B^5)$. More generally $B^{2n} = B(B^n)$ and $\therefore B^{2n+1} = B(B^n)B$. It follows that if B^n is not disjoint from A and if n is the least number for which this is true, then n is odd.

Can we show that every B^n with n odd is disjoint from A? I believe I have done this, however the proof is not very interesting. On the way to this result, however, I find a theorem giving the general form of all wffs in B^n, for each n. I shall give this result.

Form Theorem Part I. Any wff $\in B^{2^n}$ is of form either $\underline{xyn....321}$
$(yz \to xz)n....321$

(if n is even) or $\frac{(yz \to xz)n\ldots321}{xyn\ldots321}$ (if n is odd). I have written these in "fractional form". Association is to the left, arrows have been omitted, and numbers are used as schematic letters. The antecedent is given on top; the consequent on the bottom.

Pf.: Form theorem Part one is true by computation for the first few cases. I do an induction, first from even n to odd n+1. Let $\alpha = \frac{x\ y\ n\ldots321}{(yz\ xz)n\ldots321}$ be in B^{2^n}. For any $\beta \in B^{2^{n+1}}$ there is such an α such that $\beta = b\alpha\alpha$, $b \in B$. This so because $B^{2^{n+1}} = B(B^{2^n})$. Now $b = \alpha \to \beta = (w \to v) \to ((v \to u) \to (w \to u)) =$ $(\frac{x\ y\ n\ldots321}{(yz\ xz)n\ldots321}) \to ((\frac{(yz\ yx)n\ldots321}{u}) \to (\frac{xyn\ldots321}{u})$, so that $\beta = \frac{(yz\ xz)n\ldots321u}{x\ y\ n\ldots321u}$ and re-lettering gives the result. The proof from odd n to even n+1 is similar. Q. E. D.

We note that B^{2^n} is disjoint from A. If β above were $\in A$, (yz xz) would = xy, which is impossible. (This is not a very exciting argument, since $B^{2^{n+1}} = B(B^{2^n}) = \ldots\ = B(B(B(\ldots(B))))$ gives a basis for the same result without computation of the form.)

The second part of the Form Theorem is a little complicated and so I shall first illustrate with an example. The content of the theorem is essentially that the general form for all $\alpha \in B^n$ can be determined quickly from the knowledge of n's binary expression. In binary notation, 27 is $\overline{11011}$. So $B^{27} = B^{\overline{1101}}$. Our theorem tells us to first write $\frac{xy}{xy}$. Next write the binary form of n above this figure, thus $\frac{1\ 1\ 0\ 1}{xy}$. Next number the digits from the right: $\frac{5\ 4\ 3\ 2\ 1}{xy\ 1\ 1\ 0\ 1}$. Wherever the digit is '0', drop its number both below and above the line: $\frac{5\ 4\ 3\ 2\ 1}{xy\ 1\ 1\ 0\ 1\ 1}$. Next, starting from the right, drop the remaining numbers into exponent positions either on top or on bottom, depending upon the rule: (1) The

natural position of an odd number is on the bottom, for an even on the top; and (2) a number's actual position is the same as its natural if it is preceded (proceeding from the right) by an even number of exponents (both up and down, but not counting the number itself), and opposite if by an odd number. The final result of applying this rule is $\frac{xy^{54}3}{xy\ 3^{21}}$. This is the "semi-explicit form" of B^{27}.

To get the explicit form (fractional), we apply association to the left and the rule $\alpha\beta^{\gamma} = (\beta\gamma)(\alpha\gamma)$. Thus $B^{27} = \frac{[(X54)(Y54)]3}{(XY21)(321)}$. This is a great saving of work, compared to the 27 computations needed by the method used for C^1 to C^{12}.

Form Theorem. The rules just explained work for every n.

Pf: Part I. The work for $B^n = B^{2^k}$ since $2^k = \underbrace{100\ldots00}_{k\ 0's}$.

Part II. Given that they work for $n = 2^k$, they work for $n = 2^k+1$, $B^{2^k+1} = (B^{2^k})B$ and computation of α o b, $\alpha \in B^{2^k}$, $b \in B$ gives the result.

Let $2^k + 1 < n < 2^{k+1}$ and suppose the theorem proven up to and including n - 1. So if n is even, the theorem is true for (n/2) and if n is odd, it is true for (n - 1/2). Now $B^{evn} = B(B^{ev/2})$, while $B^{odd} = B(B^{ev - 1/2})B$, and these equations give the result. I leave the computations to the reader. Q. E. D.

With the Form Theorem in hand it is possible to show that every B^k is disjoint from A, at least if I am not mistaken. However I have not worked the proof together and it is in any event a piece of very complicated brute-force induction. Further in looking around to see how this result effect the original problem, I have found it does not seem quite as strategic as I'd hope it might be. It's possible that I might

be able to solve the original problem by pushing this method a bit further, but it seems likely that a matrix would be much easier to find. Further, of the results given here, the first main result was much easier to arrive at than the second (the Form Theorem) and that in turn easier than the one showing all B^n disjoint from A. It appears therefore that less and less interesting results are coming by more and more brute force, and that the present method, though usable in a pinch, is not the best for the remaining problems.

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Latest on a Logic Problem

Problem:

Given the system:

1. $\alpha \to \alpha$
2. $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
3. $(\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

Rules: MP, substitution.

To Show: There are no α, β, $\alpha \neq \beta$, such that $\alpha \to \beta$ and $\beta \to \alpha$ are both provable in the system.

We define A as the class of all instances of axiom 1, B as the class for 2, and C for 3. If X, Y are classes of formulae, then X o Y shall be the class of all formulae ν such that there is some $\alpha \to \gamma$ in X and α in Y. Thus X o Y is the modus ponens product of X and Y; it is the class of γ gotten by using MP with implications from X and their antecedents from Y.

We associate to the left and omit the dots and list some straightforward results:

0b) $BA \subset A$

1) $AX = X$

2) $BXA \subset X$

3) $CXY = BYX$

4) $BXYZ = Y(XZ)$

0c) $CA \subset A$

5) $CXA \subset X$

6) $CXYZ = X(YZ)$

Pf. 0b) BA is the class of instances of $(\alpha \to \gamma) \to (\alpha \to \gamma)$.

1) obvious. Using $\frac{\alpha}{\beta}$ for $\alpha \to \beta$, we have roughly $\frac{\alpha}{\alpha} \circ \alpha = \alpha$ $\quad \alpha \to \alpha$ $\quad (\frac{\alpha}{\therefore \alpha}$ MP $)$

2) Sketchily: $B \frac{\alpha}{\beta} \frac{\beta}{\beta} = \frac{\alpha\beta}{(\beta\beta)(\alpha\beta)} \frac{\alpha}{\beta} \frac{\beta}{\beta} = \frac{\beta\beta}{\alpha\beta} \frac{\beta}{\beta} = \frac{\alpha}{\beta}$
$\quad B \qquad\qquad X \quad A \qquad\qquad X$

Here 'X' is really the implications in X. So

$BXA = X \cap \{\text{implications}\}$

3) $CXY = C \frac{\alpha}{\beta} \frac{\nu}{\epsilon} = \frac{\alpha\beta}{(\gamma\epsilon)(\gamma\beta)} \frac{\alpha}{\beta} \frac{\nu}{\epsilon}$ and we must have $\epsilon = \alpha$ and this product $= \gamma\beta$.
$\quad \alpha \quad C \qquad X \; Y$

$BYX = B \frac{\nu}{\epsilon} \frac{\alpha}{\beta} = \frac{\nu\epsilon}{(\alpha\beta)(\gamma\beta)}$ and again $\alpha = \epsilon$ and the product is $\gamma\beta$.
$\quad \epsilon$

4) $BXYZ = B \frac{\alpha}{\beta} \frac{\beta}{\gamma} \alpha = \gamma$

$Y(XZ) = \frac{\beta}{\gamma} (\frac{\alpha}{\beta} \alpha) = \gamma$.

These proofs are just sketches. Details left to reader. 0c, 5, 6 are like 0b, 2, 4. Q. E. D.

Theorem. Every theorem of 1-2-3 which is not an instance of axiom 1 is a theorem of 2-3, that is, of the system:

2. $(\alpha \to \beta) \to ((\beta \to \nu) \to (\alpha \to \gamma))$

3. $(\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

Rules: MP, substitution

Pf. We show that every A, B, C product is either a subset of A or is a subset of some B, C product.

If an A, B, C product expression contains an A but is not A, it can be shortened (to an expression for the same or a superset class) by use of 1, then 0b, 0c, then 2, 5 and then 4, 6, depending on what the part involving A looks like. Q. E. D.

Discussion. In the original problem, the $\alpha \to \beta$ and $\beta \to \alpha$ sought are not

instances of axiom 1, since $\alpha \neq \beta$. Therefore they are theorems of 2-3 alone. But then by $(\alpha \to \beta) \to ((\beta \to \alpha) \to (\alpha \to \alpha))$ some instance of axiom 1 would be provable in 2-3. In other words some pure B, C product would contain a member of A. Thus to solve the original problem it suffices to prove every B, C product to be A-disjoint.

We next attempt to bring some order into our problem by finding some standard forms for B, C products.

Definition. B and C are left-associated B, C products. If X is, so are XC and XB. Thus lefted B, C products are simply strings of B's and C's without parentheses.

Result Any B, C product may be expressed as a left-associated one.

Pf. We define the length of a given B, C product expression as the number of B's and C's in it.

Lengths one and two are already defined.

Length 3: X(YZ) = CXYZ

Length 4: (α) X(YZU) = CX(YZ)U = B(YZ)XU = CBYZXU

(β) X(YZ)U = length 3 o U

(γ) X(Y(ZU)) = CXY(ZU) = C(CXY)ZU = (α)ZU

(ε) (XY)(ZU) = C(XY)ZU = length 3 o Z o U.

General case. Let a small s indicate a simple (B or C) term and small c a complex but lefted term.

(α) $X_s(Z_cU_s) = BZ_cX_sU_s$ = less length o X_s o U_s.

(β) $X_c(Z_sU_s) = CX_cZ_sU_s$ = (α) o Z_s o U_s.

(ν) $X_c(Z_cU_s) = CX_cZ_cU_s$ and letting $CX_c = VS$ by α, this = VSZ_cU_s and VSZ_c is again of type ν but with shorter second factor (= Z). Q. E. D.

Thus every B, C product is leftable. But more nested forms are easier to work with, so we look for a more nested standard form.

Nesting shall be resisted by the following formation:

Definition. $\underset{\sim}{D}XY = B(BX)(CY)$

Definition. B is $\underset{\sim}{D}$-nested. C is. If X and Y are, so are BX, CX, BXB, BXC, CXB, CXC, $\underset{\sim}{D}XY$, $\underset{\sim}{D}XYC$, $\underset{\sim}{D}XYB$.

In order to prove that every B, C product is $\underset{\sim}{D}$-nested, we need the following straightforward results:

7) $B(BX)(BY) = B(BYX)$

8) $B(CX)(CY) = C(BXY)$

9) $B(BX)(CY) = \underset{\sim}{D}XY = B(CY)(BX)$

10) $B(BXY)W = BX(BYW)$; $C(CWY)X = CW(CYX)$.

Pf. Directly from original definitions of B and C. Q. E. D.

Further results:

d1) $B(BV)(\underset{\sim}{D}ZU) = \underset{\sim}{D}(BZV)U$

d2) $B(CV)(\underset{\sim}{D}ZU) = \underset{\sim}{D}Z(BVU)$

d3) $B(\underset{\sim}{D}UV)(\underset{\sim}{D}XY) = \underset{\sim}{D}(BXU)(BUY)$

d4) $B(\underset{\sim}{D}UV)(CX) = \underset{\sim}{D}U(BVX)$

d5) $B(\underset{\sim}{D}UV)(BX) = \underset{\sim}{D}(BXU)(V)$

d6) $B\ulcorner\underset{\sim}{D}XZY\urcorner W = \ulcorner\underset{\sim}{D}X(BZW)\urcorner Y$

d7) $BW(\underset{\sim}{D}XYZ) = (\underset{\sim}{D}(BWX)Z)Y$

d8) $B(BXZ)Y = \underset{\sim}{D}XYZ$

d9) $BX(BYZ) = \underset{\sim}{D}XZY$.

The results 7, 8 show a certain tendency for nesting. For instance 7 takes B(BB)(BY) into B(BYB). Results d1 - d9 may be used to bring $\underset{\sim}{D}$ operators to the outside of formulae. Perhaps a sketch of the proofs would

be helpful.

Pf. $B\frac{\alpha}{\beta}\frac{\beta}{\gamma} = \frac{\alpha}{\gamma}$. Thus $B(B\frac{\alpha}{\beta})(B\frac{\gamma}{\epsilon}) = B(\frac{\alpha u}{\alpha u})(\frac{\epsilon v}{\gamma v}) = B(\frac{\beta u}{\alpha u})(\frac{\alpha(=\epsilon)u}{\gamma \quad u}) = \frac{\beta u}{\gamma u} =$

$B(\frac{\alpha(=\epsilon)w}{\gamma \quad w}\frac{\alpha}{\beta}) = B(\frac{\alpha\beta(=w)}{\gamma\beta}) = B\frac{\alpha}{\beta} = \frac{\beta u}{\gamma u}$ proves 7. Similarly 8 - 10.

For the d1 - d9, note $D\frac{\alpha}{\beta}\frac{\gamma}{\epsilon} = B(B\frac{\alpha}{\beta})(C\frac{\gamma}{\epsilon}) = B(\frac{\beta u}{\alpha u})(\frac{v\gamma}{v\epsilon}) = B(\frac{\beta\gamma}{\alpha\gamma})(\frac{\alpha\gamma}{\alpha\epsilon}) = \frac{\beta\gamma}{\alpha\epsilon}$.

Thus d1 is $B(B\frac{\alpha}{\beta})(D\frac{\gamma}{\epsilon}\frac{f}{g}) = B(\frac{\beta u}{\alpha u})(\frac{\epsilon f}{\gamma g}) = B(\frac{\beta \quad f}{\alpha(=\epsilon)f})(\frac{\alpha f}{\gamma g}) = \frac{\beta f}{\gamma g}$,

while $D(B\frac{\gamma}{\epsilon}\frac{\alpha}{\beta})\frac{f}{g} = D(B\frac{\gamma}{\alpha}\frac{\alpha}{\beta})\frac{f}{g} = D\frac{\gamma}{\beta}\frac{f}{g} = \frac{\beta f}{\gamma g}$ and the others are similar. Q. E. D.

Proofs with clearer class versus member distinctions are left to the reader.

We now prove:

Theorem: Any B, C product is expressible as D-nested.

Pf. DC XY = DXYC and DB XY = DXYB. Count D as of length 3 since DXY = B(BX)(CY), but count DS as also of length 3 since DXYS = B(BXS)Y = DS XY.

Not using D, DS, reduce every product to shortest form. Then, by 4, 6, it is B, C, BX, CX, BXY, CXY. But B, C, BX, CX are no problem and CXY = BYX. So shortest counterexample to theorem is BXY. Suppose X, Y are now in required form without lengthening. Then same for BXY.

B(CZS)Y = C(BZY)S shorts to BZY.

BY(BZS) = B(BYZ)S to BYZ.

B(BYS)Y = DXYS = DS XY involves no lengthening.

BX(CZS) = DXZS = DS XZ.

So neither X nor Y is of form SUS.

If either is of form DXYS, use d6, d7.

If of form DXY, sue d1 - d5.

If either is S, use BSX = CXS.

if both of form SV, use 7, 8, 9. Q. E. D.

We will now set aside DXY and DXYS and concentrate on trying to show

A-disjointness for purely nested products.

We define: B and C are nested; if X is, so are BX, CX, BXB, CXB, BXC, CXC.

We introduce a new notation for nested products. Associating to the right in the new notation:

BX = B(X) [B...$\overset{\text{new}}{\downarrow}$ = B(...$\overset{\text{old}}{\downarrow}$)]

CX = C(X)

ØX = B(X)B [ØX = (B(X))B]

ŁX = B(X)C

$\underset{\sim}{B}$X = C(X)B

ØX = C(X)C.

Thus every nested product is a parenthesis-free string of Ø,Ł,$\underset{\sim}{B}$,Ø,B, C ending in either B or C.

By, for instance, a Ø, B product, we shall mean a parenthesis free string of Ø's and B's ending in B. (Excuse use-mention confusions!)

We now turn to proving A-disjointness results.

This is the difficult part of our work. The above combinatory stuff is elegant and nice and is correct. The A-disjointness proofs are messy and brute-force-ish and somewhat suspect. Especially is the very end of each proof to be viewed with suspicion, since here is where I have in the past found my "proofs" to be mistaken.

Ø, B Theorem. (This theorem will be a paradigm for all the others, so its proof should be studied carefully).

Ø, B products are A-disjoint.

Pf. First some notation. I write $\alpha \overset{\sim}{\sim} \beta$ to mean that the probelm of α's A-disjointness reduces to that of β's. In other words, $\alpha \overset{\sim}{\sim} \beta$ means that α is at least as A-disjoint as β, or that if β is A-disjoint, so is α.

Further, while nested products in nested notation are associated to the right, fractional forms are associated to the left.

To work.

$BX \tilde{\sim} X$, since $B\frac{\alpha}{\beta} = \frac{\beta u}{\alpha u}$. B, $\not{\beta}^{n}B$ are bottomheavy since $B = \frac{xy}{yz\ xz}$ and $\not{\beta}\frac{x}{yz} = \frac{x}{(z1)(y1)}$. "Bottomheavy" means that every instance has a longer component (bottom). Now $\not{\beta}^{ev}BX = \not{\beta}^{ev}B\frac{x}{y} = \not{\beta}^{ev}\frac{yu}{xu} = \frac{xu}{(x\ ev\ldots\ 1)(u\ ev\ldots\ 1)}$ and this is out since u is self-included. That is, if yu were the same as (x ev... 1)(u ev... 1) then u would be a proper part of itself.

So our product (counterexample to theorem) must be

$$\not{\beta}^{od}BX = \not{\beta}^{od}B\frac{x}{y} = \not{\beta}^{od}\frac{yu}{xu} = \frac{yu}{(u\ od\ldots\ 1)(x\ od\ldots\ 1)} = \frac{y}{x\ od\ldots 1\ od\ldots\ 1}.$$

This last is the "reduced form".

Initial $\not{\beta}^{od}BX \tilde{\sim} \frac{y}{x\ od\ldots\ 1\ od\ldots\ 1}$

2od group

Counting places from the right, the Initial $\not{\beta}^{od}$ group deposits 1, ..., od, 1, ..., od in the last 2 od places on bottom and it deposts nothing on top. The B deposits nothing (its u was eliminated in reduction) but it turns $\frac{x}{y}$ upside down.

Letting the block be the next $\not{\beta}^{od}BX$ or last B or $\not{\beta}^{n}B$, we develop the pre-block part of our product.

We have:

free B term

...BX rightside up = $\frac{yu}{xu}$

...$\not{\beta}^{ev}$BX right = $\frac{y\ u}{x\ ev\ldots\ 1\ \underbrace{(u\ ev\ldots\ 1)}}$

free trail terms

condensation, including condensed B term and condensed trials

If u on bottom not in greater place (from right) than on top (either free u or condensed), have semi-block. We assume pre-semiblock.

Now each B turns later stuff over, so we also have:

...BX upsidedown = $\frac{x\ u}{y\ u}$ / previous part longer on bottom by non-semiblocking

...$\emptyset$BX up = $\frac{x\ ev...\ 1\ (u\ ev...\ 1)}{y\ u}$

These schema are deduced from original definitions of B and of $\emptyset$.

Example: $\emptyset_1\emptyset_2\emptyset_3B_4\emptyset_5\emptyset_6B_7B_8B_9\emptyset_{10}\emptyset_{11}\emptyset_{12}\emptyset_{13}B_{14}B_{15}\emptyset_{16}\emptyset_{17}B_{18}\emptyset_{19}B_{20}B_{21}$ = $\emptyset_1\emptyset_2\emptyset_3B_4/\ \emptyset_5\emptyset_6B_7/\ B_8/\ B_9/\ \emptyset_{10}\emptyset_{11}\emptyset_{12}\emptyset_{13}B_{14}/\ B_{15}/\ \emptyset_{16}\emptyset_{17}B_{18}//\ \emptyset_{19}B_{20}B_{21}$, where $\emptyset_{19}B_{20}$ is a block since $\emptyset_{19}$ is an odd number (one) of B's. Using numbers as schema variables, we may write the pre-block part as:

18 / 15 / 13 12 11 10 (14 13 12 11 10) / 14 / 9 / 8 7 / 8 6 5 (7 6 5) / 3 2 1 3 2 1

rightside up because initial plus 3 more B's is 4 B's (even)

upside down because of initial B.

block upside down / 17 16 (18 17 16)

Putting places on top above the corresponding ones on bottom:

16 15 13 12 11 10 (14 13 12 11 10) 9 / 18 15 14 9 8 / 8 7 / 6 5 (7 6 5) / 3 2 1 3 2 1

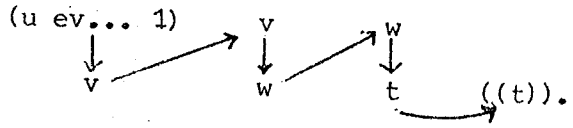
i. e. $\frac{x}{y} = \emptyset_{19}B_{20}B_{21}$ is the block, and the above fraction is not semiblocked.

Returning to the general case. Each member of the 2od group is once in odd place, once in even. Shred free u's are in same parity place top and bottom (because BX deposits top and bottom while $\emptyset^{ev}$BX deposits odd places on each side and the initial group deposited zero = even on top and 2od = even on bottom). A condensation is in the same parity place as the free version of its u. Every free trail is matched by a condensed trail on the same side ("side" = top or bottom). Most important: every top condensation is in odd, every bottom condensation in even place, so no top condensation is above a bottom condensation.

We wish now to render our so-far developed fraction bottomheavy on the assumption that it is in A (by reductio). Bottom is so far ahead by 2od group and by 2 copies of each bottom trail. The top has 2 copies of each top trail extra. One copy of these is in a condensation.

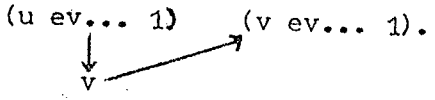
We replace extra stuff on bottom with 2 copies of each top condensation. This balances the extra stuff on top while still leaving extra on bottom since the top condensation brings down its unneeded u term.

Start with the first (rightmost) top condensation and bounce as far as necessary in odd places as follows:



Here v and w are free B terms and t is either a bottom trail term or a 2od group member. Then ((t)) is the condensed replica of t or it is the even parity copy of 2od group t. Replace both t's by (u ev... 1).

Second and further condensations are the same except for possible piggybacking:



In this case replace the bottom <u>replicas</u> of v inside (v ev... .) with (u ev... 1).

Thus either directly or by piggybacking, every top condensation in this part finds an odd place bottom t and is further rapped in either its even place copy or its condensed copy inside an even place condensation. Tracing back along the odd chain brings us back to the condensation in question, so there are no conflicts of different condensations trying to get the same t. Non-semiblocking guarantees a straightforward trend of chains.

Now come to block or semiblock. Since so far developed part is bottomheavy by above operations, semiblock is out, since bottom must be longer to avoid A-disjointness.

So we come to the block. The case of the "lasts" is trivial, since they just add what amounts to a findal condensation.

So block is $\not{B}^{od}$BX up or right.

$\not{B}^{od}$BX up = $\dfrac{u\ od\ldots\ 1\ (x\ od\ldots\ 1)}{y\quad u}$ old part

and final condensation takes everyting else (x) down and finishes problem.

So block is $\not{B}^{od}$BX right:

block $\not{B}^{od}$BX right = $\dfrac{y\quad u}{u\ od\ldots\ 1\ (x\ od\ldots\ 1)}$

We now need to bring down the condensation in y to finish our problem. Those over old parts go down as usual. There is none over (x od... 1) by parities. Those over od... 1 are absorbed. But part of y over w represents problems. This is reflected up to top u which is in even place.

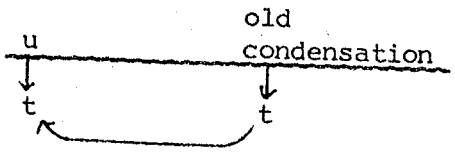
Bouncing along, this may simply be absorbed by an unused trail or 2od member. But instead we assume trouble, namely (1) we come to bottom condensation or (2) we come to used member of 2od group.

Suppose we come to bottom condensation:

$\dfrac{u}{(v\ ev\ldots\ 1)}$ In this case, condensation in y(u) wants in to v (bounce along) or into trail terms. By parities,

these are unused and so may be taken both in and out of the bottom condensation.

So we come to a 2od member. Via its other copy, this 2od member has already been used by a previous condensation. This condensation may have been part of the pre-block part or it may have come from a pre-u part of y.



We may take the v of the old condensation in both replicas. But we need some of its trails for condensations in u. By parities, the top free copies of these trails are in odd places. So bring down as if they so many extra condensations unless they were over u. If they were over u, this moves our problem rightward in u, and repeat operation. This finishes our problem, except for the lasts.

final B right = $\frac{x\ y}{yz\ \ xz}$ presents nothing extra on top.

$\emptyset$B right = $\frac{x\ y}{(?\ z\ n...\ 1)(?\ z\ n...\ 1)}$ nothing

B up = $\frac{y\ z\ \ (x\ z)}{x\ y}$ treat x z as a condensation

$\emptyset^n$B up = $\frac{(?\ z\ n...\ 1)(?\ z\ n...\ 1)}{x\ y}$ each ? is either x or y. Treat rightmost (? z n... 1) as a condensation and you are through.

So all $\emptyset$, B products are A-disjoint. Q. E. D.

Those of you who got my earlier mimeo on this problem some years ago may recall that I announced without presenting the proof that all B^k products were A-disjoint. In lefted notation $\emptyset$, B products are B^k's. This is because, mixing notations, $B^{2k} = B(B^k)$ and $B^{2k+1} = B(B^k)B = \emptyset(B^k)$.

Nonetheless what I said then was wrong in two respects. Firstly, my "proof" of the B^k theorem was defective toward the end, mainly because I had not isolated for special treatment the block and semiblock. In fact much of my time since the last mimeo has been spent writing and re-writing this one proof, trying to get the bugs out. I believe I wrote about thirteen fallacious versions of it.

I confess I am very confident of the present version. But I am not

so confident of further A-disjointness proofs I shall be giving and the reader accepts them at his peril!

Second I said there that the B^k theorem seemed unimportant because too atypical. My reason was that B^k products nest nicely and I had convinced myself that nesting was very unusual. At that time I was working with combinatory formulae 1 - 6. However my later finding 7, 8 etc. has restored my confidence in the generality of nesting (despite $\underset{\sim}{D}$).

Nevertheless extending the $\emptyset$, B proof to other cases is very slow work as the reader will see by the paucity of further results given here.

Well, let us consider what should be done next. Our bracers are:

$$B\frac{x}{y} = \frac{yu}{xu}$$

$$C\frac{x}{y} = \frac{ux}{uy}$$

$$\emptyset\frac{x}{y\ z} = \frac{x}{z1\ y1}$$

$$\mathbb{E}\frac{x}{y\ z} = \frac{x}{1y\ 1z}$$

$$\underset{\sim}{B}\frac{xl\ yl}{z} = \frac{y\ x}{z}$$

$$\emptyset\frac{lx\ ly}{z} = \frac{x\ y}{z}.$$

Now final $B = \frac{x\ y}{yz\ xz}$ and $C = \frac{y\ z}{xy\ xz}$ are bottomheavy, and only BX relieves bottomheaviness (by turning $\frac{x}{y} = X$ over). Thus our product must contain B (and at least one non-terminal B). So we do not need to prove a $\emptyset$, C theorem.

Notice further that, for instance, $\mathbb{E}^n B\frac{x}{y} = \mathbb{E}^n \frac{yu}{xu} = \frac{y\qquad u}{((n(...(1\ x))))((n(...(1\ u))))}$ has u self-included, because $\mathbb{E}$ does not switch the bottom yz around. This sort of consideration makes a $\mathbb{E}$, B theorem or a $\emptyset$, B theorem unnecessary. Thus it is natural next to (1) try the other main bracer and try a $\underset{\sim}{B}$, B theorem; (2) introduce C and try a

ϕ, B, C theorem; or (3) try a minor bracer and a ϕ, $\mathbb{E}$, B theorem.

We shall first do the second of these.

ϕ, B, C Theorem.

ϕ, B, C products are A-disjoint.

Pf. In the previous proof, places were numbered from the right. This place-numbering performed two functions: (1) the numbering ordered items so that we worked gradually leftward and bounced new condensations rightward; (2) the numbering partitioned things into odd and even and thus separated top condensation chains from bottom condensations.

In this proof we shall accomplish ordering by taking places as they are suggested by a certain re-writing procedure.

Partitioning will be accomplished separately by considering antecedentality and consequentiality re sides. Consequentiality re a side will be called "oddness", and antecedentality "evenness".

The inventor of these concepts is cited in Anderson and Belnap's book on entailment. The definition is: the consequent of a formula is consequental, the antecedent antecedental. If x is consequental, its consequent is also and its antecedent is antecedental. If x is antecedental, its antecedent is consequental and its consequent is antecedental.

We are going to figure these separately re top and bottom of our (reduced) fraction, rather than re the whole fraction. We will call them "parities".

Well, to work.

$CX \tilde{\sim}' X$. $BX \tilde{\sim}' X$. $\phi^n B$, B, C, $\phi^n C$ are bottomheavy. $\phi^{ev}BX$ out. $\phi^{ev}CX$ out.

$\phi^{od}CX = \phi^{od}\frac{ux}{uy} = \frac{ux}{(y\ od\ldots\ 1)(u\ od\ldots\ 1)} \tilde{\sim} \frac{x}{y\ od\ldots\ 1\ od\ldots\ 1}$. But $\phi^{od}BBX$ reduces to the same thing. So start as before with:

initial $\not{\beta}^{od}BX \simeq \frac{y}{x\ od...\ 1\ od...\ 1}$.

We write $\alpha\beta = \frac{\beta}{\alpha}$ in certain cases. We suppose no semiblocking and we replace the concept of block by that of non-alignment. Non-alignment is when an uncerlined part is placed over a non-underlined part or vice-versa. (So that the nth place on top does not correspond to that on bottom.) We assume alignment.

Then we have (regular means non-underlined, ✓ means underlined):

(1) BX right = $\frac{y\ u}{x\ u}$

(2) CX right = $\frac{x\ \underline{u}}{y\ \underline{u}}$ ✓

(3) $\not{\beta}^{ev}BX$ right = $\frac{y\quad u}{x\ ev...\ 1\ (u\ ev...\ 1)}$

(4) $\not{\beta}^{ev}CX$ right = $\frac{x\quad \underline{u}}{y\ ev...\ 1\ \underline{(u\ ev...\ 1)}}$ ✓

(5) $\not{\beta}^{od}BX$ right = $\frac{y\quad u}{x\ od...\ 1\ \underline{(u\ od...\ 1)}}$

(6) $\not{\beta}^{od}CX$ right = $\frac{x\quad \underline{u}}{y\ od...\ 1\ (u\ od...\ 1)}$ ✓.

The right terms (u's and (u...1)'s are all even regular or ✓ odd.

And:

(7) BX up = $\frac{x\ u}{y\ u}$

(8) CX up = $\frac{y\ \underline{u}}{x\ \underline{u}}$ ✓

(9) $\not{\beta}^{ev}BX$ up = $\frac{x\ ev...\ 1\ (u\ ev...\ 1)}{y\ u}$

(10) $\not{\beta}^{ev}CX$ up = $\frac{y\ ev...\ 1\ \underline{(u\ ev...\ 1)}}{x\ \underline{u}}$ ✓

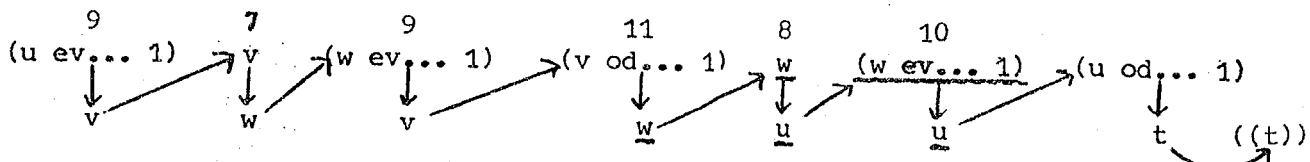
(11) $\not{\beta}^{od}BX$ up = $\frac{x\ od...\ 1\ \underline{(u\ od...\ 1)}}{y\ u}$

(12) $\not{\beta}^{od}CX$ up = $\frac{x\ od...\ 1\ (u\ od...\ 1)}{x\ \underline{u}}$ ✓.

We shall sometimes refer to odd regular and even ✓ as 'good', and to

even regular and odd as 'wrong' or 'bad'. The bad places are where bottom condensations lurk.

Looking at the above good places we see that, say, a top condensation as in 9 may bounce through 7's, piggyback on a further 9, enter through piggybacking on 11, bounce through 8's, piggyback on 10's, and get back to regular with 12. Thus:



So top condensations go down. Semiblocks out. So we come to non-alignment.

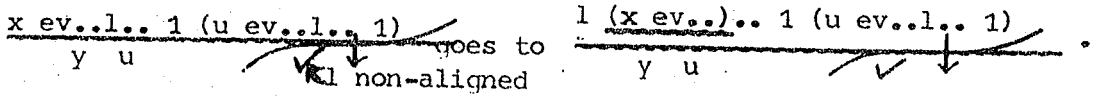
Before considering non-alignment, a few remarks about parities (these apply also essentially to ∅, B theorem). Every atom occurs twice (before our replications, of course). It occurs either twice on one side, once in each parity (2od terms or trail terms) or it occurs once on each side, both times in the same parity (u's).

Now non-alignment.

By self-inclusion, the top u of the right cases cannot be the non-aligning part. Similarly with 7, 8.

The top condensations of 9 - 12 cannot be non-aligned, for then swithcing out an odd regular, even ✓ part gives us the rest in an even ✓ odd regular (still good) condensation and bringing it down, we are through. So non-aligned must be a top trail in 9 - 12.

For instance:



The bottom non-aligned part (here the ✓) must be in the old part. If it is past u, we will have semiblocking (-like) trouble. If it were at u, all of x ev... 1 would go piggyback. (Similarly in 10 - 12.)

Since top non-aligner is as trail, it is non-underlined and below is a ✓as in our example. If it is in a good place, so is ✓and all the rest is absorbed, so it is in a bad place and so then is (x n...) after reversal. So we bounce along the bad chain and come to either a bottom condensation or an already used member of the 2od group.

First re-write (x n...) so that all condensations are odd regular. (Switch first even ✓condensation and this puts all the rest in one last odd regular condensation: x'....(____) → ____(x'....).) So all the condensations in x are odd regular.

Suppose we come to a bottom condensation. We take its odd regular trails, not used, and bounce its u term. So we come to a used member of the 2od group. A top condensation is there. It may be from the pre (u...l... 1) part or it may be (u...l...1) itself. If the former, takes its u term and then take the free top copies of its needed trails in condensed position and we're through. So let the condensation be (u...l... 1) itself. The trails ... 1 can be used. The trail 1 is in the wrong parity to be needed. The trails in (x n...) move our problem rightward and so into the ... 1 ultimately.

It remains to consider lasts. Nothing new.

Therefore all Ø, B, C products are A-disjoint. Q. E. D.

We next wish to try the effect of a minor bracer, either Ø or Ł. Since Ø, like B, removes rather than deposits parts, we shall leave this bracer until we have considered the (so far unsolved) problems raised by such removals in connection with the B, B theorem (a conjecture at this point!).

My last "proof" of the Ø, Ł, B "theorem" is defective at the end at this point in time. However we shall attempt it here, finishing it if I can straigthen out the bugs.

Let us start bravely by announcing:

Ø, Ɛ, B Theorem.

All Ø, Ɛ, B products are A-disjoint.

Pf. $Ø\frac{x}{y\ z} = \frac{x}{z1\ y1}$

$Ɛ\frac{x}{y\ z} = \frac{x}{1y\ 1z} = \frac{x}{y1\ z1}$

Let $\overline{ØƐ}^{od}$ be a non-empty string of Ø's and Ɛ's with an odd number of Ø's and let $\overline{ØƐ}^{ev}$ be a non-empty string of Ø's and Ɛ's with an even number (maybe zero) of Ø's.

B out. $\overline{ØƐ}^{n}BX$ out. $BX \tilde{\sim} X$.

$\overline{ØƐ}^{ev}BX = \frac{y\ u}{(x\ n\ldots\ 1)(u\ n\ldots\ 1)}$ is out.

even number of them are regular

So product is: $\overline{ØƐ}^{od}BX \tilde{\sim} \frac{y}{x\ n\ldots\ 1\ n\ldots\ 1}$

Initial

odd number of them are regular

Assume alignment, no semiblock, and then:

$\ldots BX\ right = \frac{y\ u}{x\ u}$

$\ldots\overline{ØƐ}^{ev}BX\ right = \frac{y\ u}{x\ n\ldots\ 1\ (u\ n\ldots\ 1)}$ ev

$\ldots\overline{ØƐ}^{od}BX\ right = \frac{y\ u}{x\ n\ldots\ 1\ (u\ n\ldots\ 1)}$ od

$\ldots BX\ up = \frac{x\ u}{y\ u}$

$\ldots\overline{ØƐ}^{ev}BX\ up = \frac{x\ n\ldots\ 1\ (u\ n\ldots\ 1)}{y\ u}$ ev

$\ldots\overline{ØƐ}^{od}BX\ up = \frac{x\ n\ldots\ 1\ (u\ n\ldots\ 1)}{y\ u}$ od

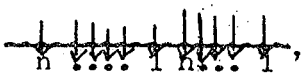
Alignment puts underlined condensations over underlined trails or 2od members for immediate replication. Underlined trails are put over same.

Other condensations go down as usual. Thus there is no semiblock and we come to non-alignment.

Non-alignment brings in some difficult problems. For this reason it will be good for us to have a firmer grasp of the aligned part.

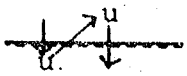
For this purpose we want to look more fully both at the good (odd regular, even ✓) chains and at the bad (even regular, odd ✓) chains. We develop these from their ends backwards to their starts (if any).

First we deposit the 2od group. Each 2od member ends a chain:



, either good or bad, depending on the type of member.

Each free B term continues a chain, of whichever type:



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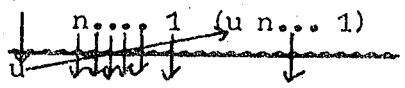
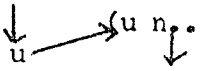
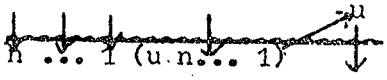
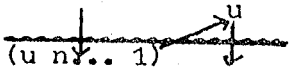
Each top condensation formation 

does two things. The u and condensed part 

continues a good chain backwards. And each free trail <u>starts</u> a chain, either good or bad, and therefore ends our backward chain-tracing.

Each bottom condensation formation 

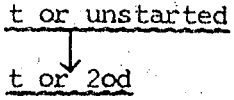
does two things. The condensed part 

continues a bad chain. The trails <u>end</u> new chains (and start us tracing new chains backwards).

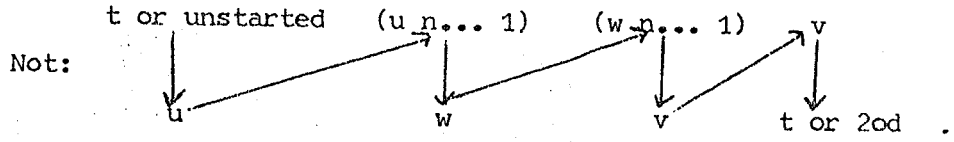
Now in developing chains backwards we may never get to the start of a given one, since the place above u or (u n... 1), say, may be yet undeveloped. Thus some chains are "unstarted".

Thus the types of chains may be schematically indicated.

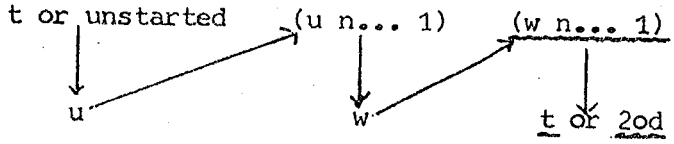
First there are the good chains. If one starts with an underlined part (trail) or is unstarted with an underlined blank, it immediately ends:



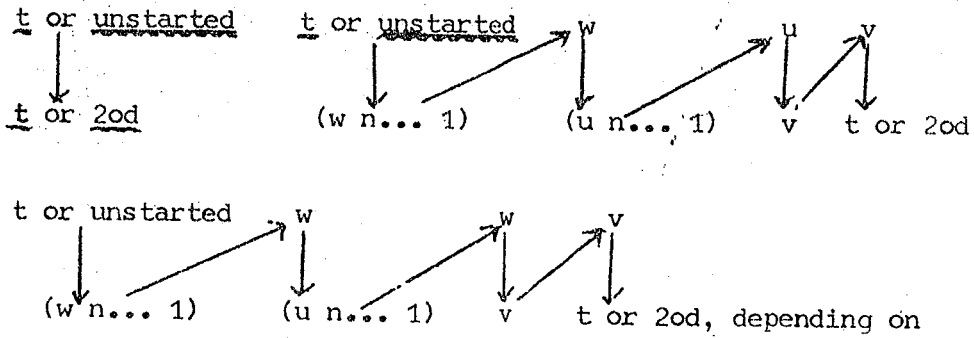
Otherwise its end is either underlined or not:



or it ends with an underlined part:



Bad



whether the bad chain ends underlined or doesn't, and if not, then whether it starts underlined or doesn't.

Now we shall after non-alignment continue to be engaged in two basic kinds of movements. A condensation or condensation-like part may be going down a good chain. It must have filled the unstarted part or it may have been bounced from a bottom replica of the starting t. In the latter event, it will come from the condensed replication (or the bad replica in the 2od group) of the condensation containing t. Alternatively, a condensation in good position or a "whole rest of top" in bad may bounce into a trail (n...1) in one of the condensations of the good chain. In either case, the part in question will be bounced to the free version of that trail. And in either case the trail was originally approached by the condensation-or-rest coming down onto the bad replica of the condensation containing the

trail (as part of the larger 'rest'). The point is that each thing can be approached only one way.

Now for bad chains. An "all rest" may come down t or unstarted. Only good trails or at most one bad trail will be wanted. Good trails will shift things to an unused bad chain. Or we may reach a bad and already used 2od group member, replicate into the u of the condensation there and shift other problems to new chains, at most one bad.

Of course this is all just a sketch. The essential point is that any given part may be first approached in one way and "re-appoached and shunted off from" in one way, so that conflicts are resolved.

Well, to work. We come to non-alignment.

The top non-aligning element cannot be a u. For instance in the BX right case, if u non-aligns: BX = $\frac{y\ u}{x\ u}$ ✓ then switching would give $\frac{u\ y}{x\ u}$ ✓ and u is self contained. Same for top u in other right cases and in BX up case.

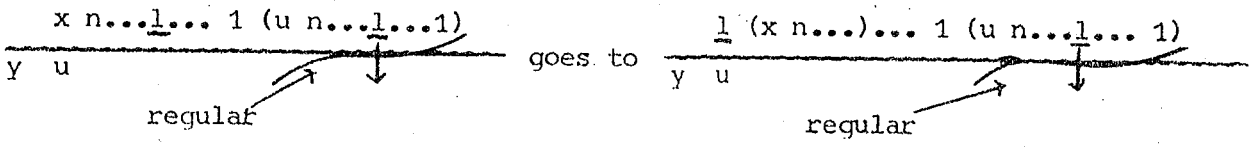
Not can it be a top condensation, for then switching puts all the rest in a condensed position and its absorption finishes problem. For instance $\frac{x\ n\ldots\ 1\ (u\ n\ldots\ 1)}{y\ u}$ ✓ goes to $\frac{u\ n\ldots\ 1\ \underline{(x\ n\ldots\ 1)}}{y\ u}$ ✓ and the thus-created condensation $\underline{(x\ n\ldots\ 1)}$ goes down (switching replaces an odd regular even ✓ with an even ✓ odd regular condensation, still good).

So the non-aligned is a trail term. If this is good, switching gives all the rest in condensed position, so this is out. So the non-aligned is a trail term in a bad position. Call this trail term 1. It is either underlined or regular. If regular:

$\frac{x\ n\ldots 1\ldots\ 1\ (u\ n\ldots 1\ldots\ 1)}{y\ u}$ ✓ goes to (ν) $\frac{1\ \underline{(x\ n\ldots)}\ldots\ 1\ (u\ n\ldots 1\ldots\ 1)}{y\ u}$ ✓

where the (u..l..1) part may be underlined or not.

If l was underlined then:



where (u..l..1) may be underlined.

In either case, since l was bad, so is (x n...). What we need now is the condensation in a. So (x n...) does down a bad chain looking to replicate its condensation.

Let us go back to pre-non-alignment momentarily. When a condensation goes down a good chain, imagine that each link of the chain was marked with a little '1' for "used in the first way". Then when the condensation is prelicated the second time (in the condensed trail or the bad copy of a 2od member), mark this with a little '2'.

Now we have (x n...) going down a bad chain and marking each link with a '1'. If we run into no problems we are finished. So we run into either a bottom condensation or a used 2od group member. In the first case, some of the condensed trails are marked '2'. We do not need these unless there is secondary non-alignment, which we defer momentarily. We may take good and unused trails and if we get to the condensed u term, we bounce on in the bad chain. Whenever we take a good condensed trail term, we mark it '1' and its free version (bad) '2'. Finally, either by non-alignment over a bottom trail in a condensation, or by arriving at a used 2od member, we find all the rest over the second replica of some top condensation. If this is the condensation (u..l..1), we call this the tangle. We defer this case. Otherwise we are coming in way one at something already filled in way two with a top condensation. Unless there is further non-alignment,

we need its good trails which are in good position in their top free versions and so we send condensations down these as yet unused chains. So assume further non-alignment in a bad place. So we reflect all the rest into a bad free trail on top and start down a new chain (bad). Etc. No conflicts develop.

The nth time we start a bad trail, we go along until coming to (1) used trail term (used in way two), (2) bottom condensation (some of whose parts are used in way two, but since we are coming at it in way one, it cannot have previously been approached this way), (3) a used 2od member (used previously in way two). In case (1) and (3) we run into a previous top condensation. In case (2) a bottom condensation. The case 2 presents nothing new. Case 1 and 3 present nothing new unless either the condensation already there is from (x...) or it is the tangle (u n..l.. 1). If it is from (x...), it is from a previously done part of it. Take good trails (moving earlier in (x...)) and replicate into the u term of the condensation in question. If further non-alignment, we find ourselves in an earlier bad place of (x...) with everything left of top. In this case, in our earlier moves with (x...), that bad place ran into an already used bottom condensed trail term or into a trail of a second replicated top condensation whose top free version is in a bad place. In either case, nothing new.

Thus we come to the tangle. Everything is coming down on (u n...l... 1). u is absorbing and l is a bad place. Unless there is non-alignment, we want good trails. The ones ...1 present no problem. The ones n... simply move trails earlier in (x n...) and ultimately bounce int ...1. Only real problem is non-alignment over l. Then all the rest is put up into top free version of l.

version of 1.

So all the rest (a part of (x...)) is in 1 with parities intact. Condensations in x over old part go down as usual and those of yu as well, unless there is a second main non-alignment.

As far as I can see, nothing essentially new. Lasts are trivial. Q. E. D.

Thus we seem to have proven that Ø, Ɛ, B products and Ø, B, C products are A-disjoint. The end of the Ø, Ɛ, B proof still needs to be more spelled out, but introduction of first and second way concept and tangle concept seems to clarify things a bit. To become confident of one of these A-disjointness proofs I have to re-write them from memory a few times and give myself a chance to become confused or worried about details. A more recursive handling of repeated main non-alignments is really needed.

Supposing that our Ø, Ɛ, B proof holds up, the next step is to put our two results together and attempt a Ø, Ɛ, B, C proof. My last version of this was defective, but the defects were inherited from the Ø, Ɛ, B proof, so if this latter is sufficiently clarified, the Ø, Ɛ, B, C proof should go through. Let us attempt this: of course a solid proof of this theorem would give all previous theorems as corrolaries, although it is easier to develop our proof concepts piecemeally.

Ø, Ɛ, B, C Theorem.

All Ø, Ɛ B, C products are A-disjoint.

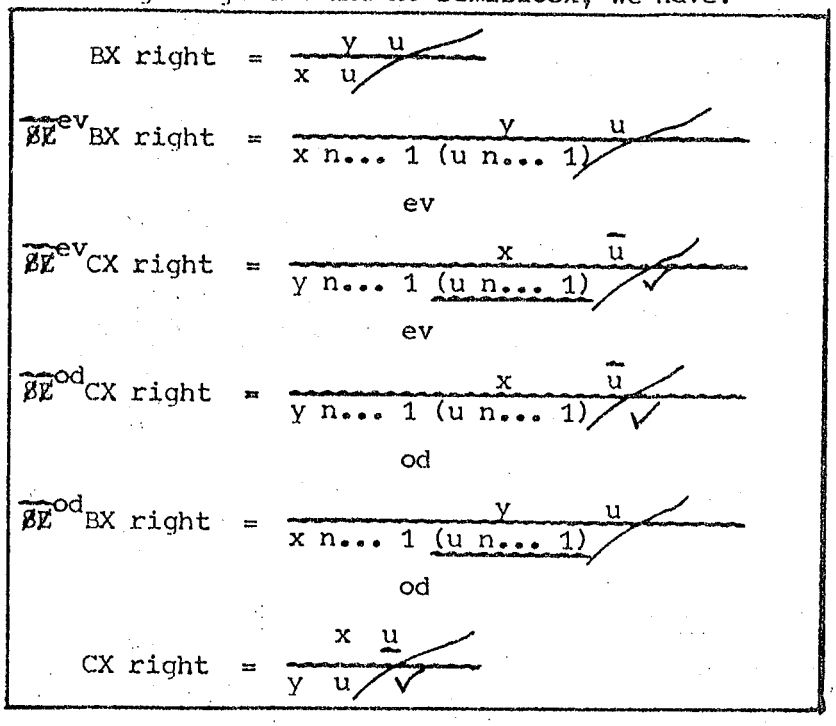
Pf. The initial part is $\overline{\text{Ø}\text{Ɛ}}^{od}{}_{BX} \simeq$ y / x n... 1 n... 1 (od)

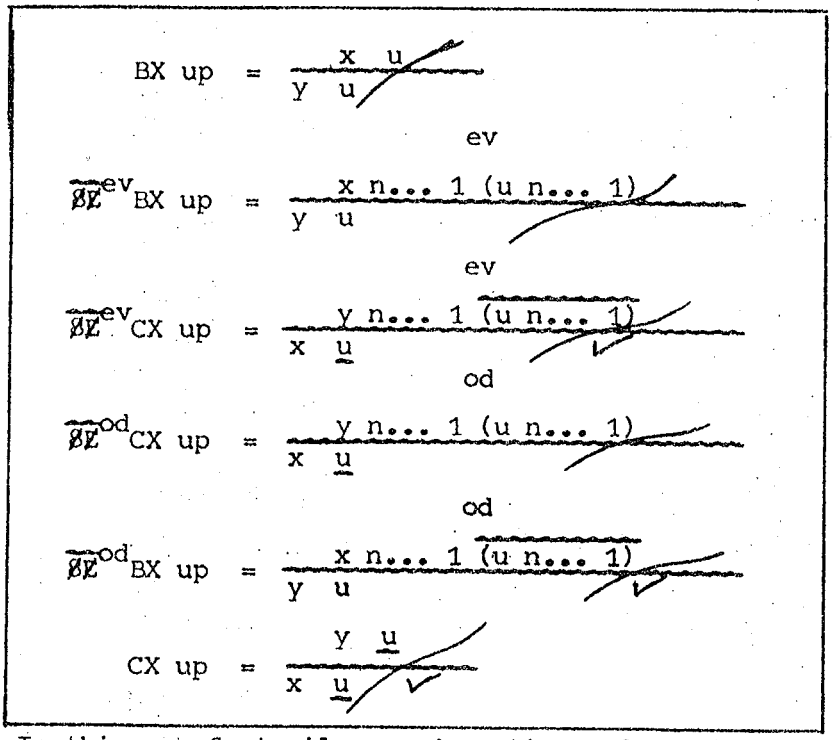
Initial

or it is $\overline{\text{Ø}\text{Ɛ}}^{od}{}_{CX} \simeq$ x / y n... 1 n... 1 (od)

The C reverses parities but is otherwise not interestingly different. So we suppose the first case.

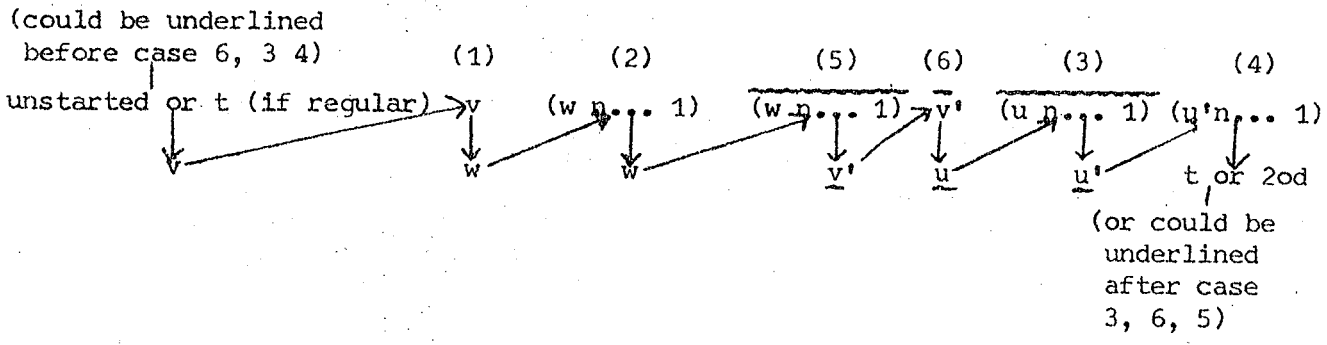
Assuming alignment and no semiblock, we have:


BX right = y u / x u
ØΣ^ev BX right = y u / x n... 1 (u n... 1)
ev
ØΣ^ev CX right = x ū / y n... 1 (u n... 1)
ev
ØΣ^od CX right = x ū / y n... 1 (u n... 1)
od
ØΣ^od BX right = y u / x n... 1 (u n... 1)
od
CX right = x u / y u

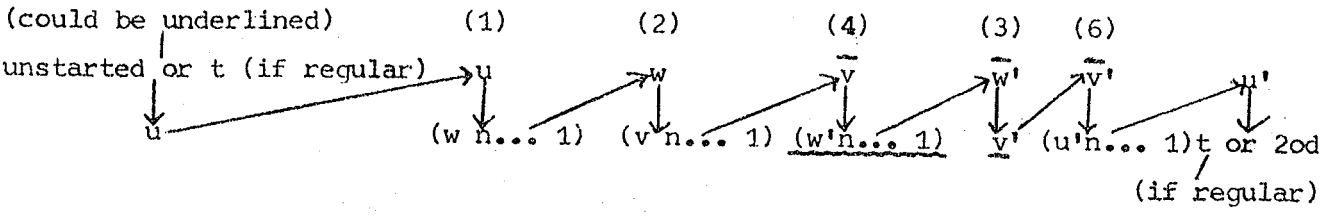


In this proof, trails, condensations, 2od members, and main terms (free C's) can be underlined.

A typical good chain looks like this:



A typical bad chain looks like this:



So no semiblock and we come to non-alignment. This can't be u or $\tilde{u}$; it can't be the condensation. It must be the top trail in the bad place.

For instance: $\overset{x\ n\ldots l\ldots\ 1\ (u\ldots l\ldots 1)}{\overset{}{y\quad u}}$ goes to $\overset{1\ \underline{(x\ n\ldots)}\ldots\ 1\ (u\ldots 1\ldots\ 1)}{\overset{}{y\quad u}}$

The rest is the same as in the previous proof. Q. E. D.

Admittedly a bit of bluffing here! The end of the previous case should be worked out more thoroughly but with the present theorem in mind and it ought to go through, but I spare myself the work and the reader the details (until my next mimeo anyway).

Supposing the Ø, Ɛ, B, C theorem (?) is proven, this means that all nested products not involving Ø or $\underset{\sim}{B}$ are A-disjoint.

The natural next step is to take the major bracer $\underset{\sim}{B}$ and try to prove a $\underset{\sim}{B}$, B theorem.

Unfortunately the post-block part of the proof presents problems I have not figured out how to handle yet (although they do not seem unsolvable; it is just that this is tedious work). Therefore I shall present only the first part of the projected proof.

B̃, B Conjecture.

All B̃, B products are A-disjoint.

Part of Proof. $\frac{\beta 1\ (\beta 1)}{y} = \frac{\beta\alpha}{y} = \frac{(\gamma 1\ (\beta 1))\ 1}{y}$.

B̃B out, $\tilde{B}^{ev}BX = \frac{yu}{xu}$ ev ... 1 = what?

In order to apply ev, call the consequent of y "1" and reletter u as u1.

Then $= \frac{y^{1}_{=}(u1)}{x(u1)}$ ev ... 1 $= \frac{y^{1}_{=}\ u}{x\ (u1)}$ ev-1 ... 1. Continuing in this way

$= \frac{y^{ev\ldots 1}_{=\ =}\ u}{x\qquad (u\ ev\ldots 1)}$ where ev...1 represent subtractions of y's first even consequents and ev...1 are holes ready to receive same. Clearly is is out by self inclusion of u.

So fraction is:

$\tilde{B}^{od}BX = \frac{y\ u^{\,od\ \ldots\ 1}}{x\ u} = \frac{u\ (y^{od\ldots 1})}{x\ (u\ od\ldots 1)} \simeq$

$\frac{y^{od\ldots 1}}{x\ od\ldots 1}$.

The effect of the initial $\tilde{B}^{od}BX$ group is to deposit od holes on the bottom and od subtractions on the top to pay them off.

Putting any further $\tilde{B}^{od}BX$ as a block (or say lasts), the pre-block forms are:

BX right $= \frac{y\ u\ \simeq}{x\ u\ \simeq}$

$\tilde{B}^{ev}BX$ right $= \frac{y^{ev\ldots 1}\ u\ \simeq}{x\ (u\ ev\ldots 1)\ \simeq}$

BX up $= \frac{x\ u\ \simeq}{y\ u\ \simeq}$

$\tilde{B}^{ev}BX$ up $= \frac{x\ (u\ ev\ldots 1)\ \simeq}{y^{ev\ldots 1}\ u\ \simeq}$

where the ≃'s remind us that parts may have been pre-subtracted for earlier holes

We see that the only actual deposits are u's. The rest are condensations with holes and the initial od group of holes.

All condensations, including those removed inside of others, are in consequential place re whole. Thus top condensations are in even, bottom in odd places re their sides (opposite from previous proofs). Otherwise parity properties hold up. (Subtractions leave parities re whole unchanged after od group. The change in the od group is offset by the turn-over of $\frac{x}{y}$.)

At any point in the development of our fraction we conceive yet-to-be-subtracted parts as belonging to the undeveloped part, but the empty holes as already developed.

Thus extra on either side is two copies of each stolen atom (If we each have a penny and I steal yours, I have two more than you) unless it has been 're-stolen', and one copy of each unfilled hole.

Thus we need as usual two copies of each top condensation on the bottom (holes and all).

But what is to be meant by 'semiblock' here? We define a semiblock as the first u, if any, whose development fails to leave the bottom longer.

It is clear that the pre-block part has its condensations go down as usual. When $\underset{\sim}{B}^{ev}$BX up $= \frac{x\ (u\ \underline{\underline{ev}}\ldots\underline{1})}{y\ \ u}$ is developed, the bottom is longer and if there is any top at all, the $\underline{\underline{od}}$ holes are filled, if only with holey condensations. Thus something, and not a condensation, is below u $\underline{\underline{ev}}\ldots\underline{1}$, unless it is subtracted. If it is subtracted, it is either put on the bottom and so does not represent a top condensation, or it is double-subtracted into a previous top condensation and is taken care of with that top condensation. Otherwise there is some v below it.

If this v is not extra on the bottom (and thus absorbing), then its top copy is to the right $\overset{v}{\underset{v}{\nearrow}}$ by non-semiblocking, unless it was double subtracted, in which case prior replications have created two extra bottom copies. Thus bouncing along we either get to something extra on the bottom or we piggyback. (Oops. Those were prior replications because all subtractions condensed things on atoms less than themselves).

So there is no semiblock and we come to the block. Suppose the block is last B, up or down.

Suppose it is up $\dfrac{yz \quad xz \quad \tilde{}}{x \quad y \quad \tilde{}}$. If there are no subtractions, xz goes down like a condensation. If there are subtractions on the bottom only, the same thing occurs. So xz is subtracted and fills some old hole. If this is on the top, it has two bottom replicas. If this is on the bottom, it has no top replicas. In either case, it occurs an extra time on the bottom. So z in yz is matched, and x is taken down an extra time.

Other lasts are eliminated similarly.

So the block is $\underset{\sim}{B}^{od}BX$ group, up or down.

$$\underset{\sim}{B}^{od}BX \text{ right} = \frac{u\ (y \underset{\sim}{od} \cdots \underset{\sim}{1})\ \tilde{}}{x\ (u \underset{=}{od} \cdots \underset{=}{1})\ \tilde{}}$$

{bottom (u...) must be subtracted to avoid self-inclusion.

$$\underset{\sim}{B}^{od}BX \text{ up} = \frac{x\ (u \underset{=}{od} \cdots \underset{=}{1})\ \tilde{}}{u\ (y \underset{\sim}{od} \cdots \underset{\sim}{1})\ \tilde{}}$$

Hereafter complications ensue. End of Partial Proof

Summary and Prospects

The original problem remains unsolved - unless someone else has solved it in the meantime. I have perhaps proven that all $\not\emptyset$, $\not\in$, B, C products are A disjoint, or at any rate that all $\not\emptyset$, B, C ones are. If the proof for $\not\emptyset$, $\not\in$, B, C can be firmed up and the problems presented by

$\underset{\sim}{B}$, B can be worked out, then it seems likely that a proof for all nested products will be forthcoming "shortly" thereafter.

Supposing all nested products were proven A-disjoint, how could would we then be to a solution of our iriginal problem, i e to a proof that all B, C products were A-disjoint?

It is possible to be quite pessimistic about this. One line of thought is that the $\underset{\sim}{D}$-nesting theorem is not much of a nesting theorem, but is more of a $\underset{\sim}{D}$-theorem. In fact many mested products can be reduced to $\underset{\sim}{D}XYS$ forms. On the other hand a more optimistic view would be that therefore some $\underset{\sim}{D}XYS$ forms were reducible to nested ones. A neutral view is that in any event, we have to develop methods of proving A-disjointness in order to solve the problem by my means, and the nested products are as good a place as any.

The problematic results just mentioned are the following, which I have just discovered:

$$\underset{\sim}{D}X(BS_1 Z)S_2 = BS_1 (\underset{\sim}{D}XZS_2).$$

In other words:

$$\underset{\sim}{D}X(\emptyset Z)B = \emptyset(\underset{\sim}{D}XZB)$$

$$\underset{\sim}{D}X(\emptyset Z)C = \emptyset(\underset{\sim}{D}XZC)$$

$$\underset{\sim}{D}X(\not{E}Z)B = \not{E}(\underset{\sim}{D}XZB)$$

$$\underset{\sim}{D}X(\not{E}Z)C = \not{E}(\underset{\sim}{D}XZC).$$

And using CS_1 for $\underset{\sim}{B}$ and $\emptyset$, we have

$$\underset{\sim}{D}(CS_1 X)ZS_2 = CS_1 (\underset{\sim}{D}XZS_2)$$

where I have written these ($\rightarrow$) in the more optomistic direction. We also have:

$$\left.\begin{aligned} \underset{\sim}{D}XS_2S_1 &= BS_2\ BS_1\ X \\ \underset{\sim}{D}S_1ZS_2 &= CS_2\ CS_1\ Z\ . \end{aligned}\right\}$$ eliminating $\underset{\sim}{D}XSS$ and $\underset{\sim}{D}SXS$ altogether

We may also remark that $\underset{\sim}{D}XY = \underset{\sim}{D}\frac{\alpha}{\beta}\frac{\gamma}{\epsilon} = \frac{\beta\ \gamma}{\alpha\ \epsilon}$ is obviously A-disjoint if either X or Y is.

The results given above come immediately from d6 - d9.

In any event, what is really to be hoped for is not more A-disjointness proofs of the very tedious sort given here, but rather some structural insights into what is making them go. Perhaps some of you logician types out there have some nice structural concepts just waiting to formulate a more abstract and elegant proof? My own efforts to take more abstract approaches have mainly given me the insight that there is no substitute for attention to the depressing gritty details. But surely there must be a more insightful approach to these problems!

Well, enough of chatting with you guys. I must get back to work on my logic problem. Well, let me see now. There are these three axioms, and we want to show that .....

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THE TWO-PROPERTY

This note establishes the result mentioned in section 8.11 of Entailment, that each theorem of the system determined by the two transitivities is a substitution instance of a theorem of that system in which each variable occurs exactly twice. Though it is doubtful that the result will be relevant to a solution of the problem of that section (Is the system T-W "minimal" in the required sense?), still, it might be.

Notation and definitions:

p, q, r, etc., range over variables.

A, B, C, etc., range over formulas.

Square brackets are used instead of subscripts, e.g., A[1], B[n].

A formula has the two-property if every variable occurs therein exactly twice.

S is a system of propositional logic with modus ponens for the --> and substitution for variables as its only rules.

THEOREM. If every axiom of S has the two-property, then every theorem of S is a substitution of another theorem of S which has the two-property.

The following obviously suffices:

LEMMA. *If* A is a substitution of A[1], where A[1] has the two-property, *and if* A==>B is a substitution of A[2]==>B[2], where A[2]==>B[2] has the two-property, *then* we can find a B[3] of which B is a substitution, where B[3] has the two-property.

By a *structural substitution* I mean the result of taking a formula A(p) containing p and uniformly substituting B for p, yielding A(B), subject to two conditions: (1) each variable in B occurs exactly once in B; (2) no variable in B occurs in A(p).

If B arises from A by a series of structural substitutions, I will say that B is a *structuring* of A.

Fact 1. If A has the two-property, so does every structuring of A.

By a *variable identification* I mean a substitution in which the substituted formula is a variable, not a complex formula; i.e., substituting q for p in A(p), obtaining A(q). This is perhaps only interesting when it is really an identification, i.e., when q appears in A(p), but that is not part of the definition.

If B arises from A by a series of variable identifications, I will say that B is an *identification of* A.

Fact 2. If B is a substitution of A, then there is a C such that C is a structuring of A and B is an identification of C. This amounts to: if B is a substitution of A, then it is possible to pass from A to B by a series of structural substitutions followed by a series of variable identifications.

Proof (uninteresting). If A' is a substitution of A, then one can get from A to A' by a series of single substitutions. Each single substitution of B for p can obviously be replaced by a structural substitution of B' for p, following by the series of variable-identifications required to get from B' to B. Now we have a series of substitutions each of which is either a structural substitution or a variable-identification. And evidently a variable-identification followed by a structural substitution can be replaced by (at most) two structural substitutions--one for the substituted-for variable, one for the substituted variable if it occurs in the original premiss--followed by a series of variable-identifications.

By <u>structural similarity</u> between A and B, I mean that A and B are exactly alike except for their variables. One way of putting this is to say: the result of putting p for all variables in A yields exactly the same formula as putting p for all variables in B.

Fact 3. If A and B have a common substitution-instance, C, then there are structurally similar A' and B' such that C is an identification of both A' and B', and such that A' is a structuring of A and B' is a structuring of B.

Proof: Fact 2.

Now let us assume the hypothesis of the lemma, using letters accordingly. Since A[1] and A[2] have A as common substitution-instance, there must by Fact 3 be formulas A[1'] and A[2']⟹B[2'] such that

(1) A[1'] is a structuring of A,

(2) A[2']⟹B[2'] is a structuring of A[2]⟹B[2],

(3) A[1'] is structurally similar to A[2'],

(4) A is an identification of both A[1'] and A[2'],

(5) A⟹B is an identification of A[2']⟹B[2'];

hence,

(6) B is an identification of B[2'].

Also, by Fact 1, (1), (2), and the hypothesis of the lemma,

(7) A[1'] and A[2']⟹B[2]' each have the two-property.

Since by (3) A[1'] is structurally similar to A[2'], their variable-occurrences are mated under a natural 1-1

correspondence. Let a, b, range over variable-occurrences, and let f be the function which maps A[1']-occurrences onto their mates in A[2'], and also maps A[2']-occurrences onto their mates in A[1']. f is 1-1, of period two, and has no fixed point. We shall say two variable-occurrences a and b are <u>similar</u> if a and b are occurrences of the same variable. Now partition the variable-occurrences in A[2'] into two classes. X: those whose similarity-mates are in A[2'] itself. Y: those whose similarity-mates are in B[2']. Let g be that function defined on both all A[1']-occurrences and on all X (but <u>not</u> on Y), which takes each variable-occurrence into its similarity-mate: a is similar to ga. Evidently g is 1-1, of period two, and has no fixed point. We will use the following features of f and g.

(8) f and g are 1-1: if fa = fb, or ga = gb, then a = b.

(9) a = fgb implies gfa = b, since f and g are of period two (ffa = a and ggb = a).

(10) Neither f nor g have a fixed point: neither fa=a nor ga=a.

(11) f is defined on all occurrences in A[1'] and A[2'] (see (3)), while g is defined on all of these except those in Y, where it is undefined (see (7)).

By an "identification-class" of a variable-occurrence we intend the class of those which must be made similar to the given occurrence in order to reach a common identification of A[1'] and A[2']. Thus, for a variable-occurrence, a, in either A[1'] or A[2'], define Id(a) as the smallest class such that

(12) $a \in Id(a)$; $b \in Id(a)$ implies $f(b) \in Id(a)$; and $b \in Id(a)$ implies $gb \in Id(a)$.

That is, Id(a) is the closure of {a} under f and g.

Define A[1*] and A[2*]⟿B[2*] as the respective results of making all and only those identifications in A[1'] and A[2']⟿B[2']. Obviously A[1*] ≡ A[2*], since f-matching variable-occurrences are the same. So B[2*] is by modus ponens a theorem of S. Also, B is an identification of B[2*], since by (6) it is so of B[2'], and no unnecessary identifications were made in getting from A[2']⟿B[2'] to A[2*]⟿B[2*]. So it suffices to show that B[2*] has the two-property. It is obvious that each variable occurring twice in B[2'] occurs twice in B[2*]; since occurrences of these variables belong to no Id(a), no substitutions are made for them at all. We therefore need concentrate only on those variables occurring exactly once in B[2'], showing that these occur exactly twice in B[2*]. From the point of view of variable-occurrences, this is to say that each variable-occurrence in B[2'] becomes identified with exactly one distinct variable-occurrence in

B[2'] during the passage from B[2'] to B[2*]. Now the similarity-mates of variable-occurrences of variables occurring once in B[2'] must be in Y, so that it suffices to show that for each Id(b) such that $b \in Y$, exactly two members of Y belong to Id(b). In order to show this, choose an arbitrary b in Y, and construct the following sequence, which exhausts Id(b):

(13) b, fb, gfb, fgfb, gfgfb, fgfgfb, ..., .

Since f is defined on b and the entire range of g, and since g is defined on the range of f outside Y, (see (11)), the only way the sequence can terminate is in a member of Y. Furthermore, since for each element of the sequence either its predecessor (if any) or its successor (if any) is its similarity-mate under g, no element which is neither initial nor terminal can be a member of Y; for g is not defined on Y (see (11)). Therefore, it is enough to show that the sequence never repeats itself, for from this it obviously follows that the sequence must terminate, the terminus being a member of Y distinct from b. Since the sequence exhausts Id(b), this will show that Id(b) contains exactly two members of Y, and will complete the proof.

Let us call the members of the sequence (10) b[1], b[2], ..., with b=b[1]. Evidently if there is any repetition we must have b[1] = b[j] (j<1), since if b[k] = b[j], then by the 1-1-ness of f and g, b[k-1] = b[j-1]. So assume for reductio that b = b[j], j<1.

Since b is in Y, it is not in the range of g, so that we must have b = ((fg)!n)fb, where ((fg)!n) signifies a sequence of n fg's. Suppose first that n is even. Now n/2 applications of (9) yields ((gf)!(n/2))b = ((fg)!(n/2))fb. Then rewriting the right side as f((gf)!(n/2))b shows f to have a fixed point, impossible (10). Now suppose n is odd. From b = ((fg)!n)fb we obtain in the same way as before, by ((n-1)/2)+1 applications of (9), the identity

((gf)!((n-1)/2)+1)b = ((fg)!(n-1/1))fb.

Then rewriting the left side as g((fg)!(n-1)/2)fb exhibits that g has a fixed point, impossible (10).*

Nuel D. Belnap, Jr.
University of Pittsburgh
September, 1976

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Computer Studies of $T_{\rightarrow}$ - W - I

J.A. Kalman

Over the past two or three years I have been developing a program which causes a computer to prove theorems in non-classical sentential logics (cf.[3]). When Nuel Belnap passed through Auckland early in 1976, he drew my attention to the conjecture that no substitution instance of $A \rightarrow A$ is derivable from the two transitivities alone [1,p.95], and suggested that it would be interesting to apply my program to generate theorems from the two transitivities. The object of this paper is to discuss the outcome of this suggestion. A general indication will be given of the basic algorithm and main heuristics employed by the program, followed by a discussion of the nature of the output when the program is applied to the two transitivities, some notes on the performance of the program, and some conjectures suggested by a perusal of the output. No counterexample was found to the conjecture from *Entailment* [1,p.95] mentioned above.

I wish to express my gratitude to Professor Belnap for suggesting this work during his visit to Auckland and for helpful correspondence since, and to him and Professor Wolf for encouraging me to submit this paper to *The Relevance Logic Newsletter*.

The program is based on iteration of an algorithm for C.A. Meredith's operator D of "condensed detachment" [8,p.318] or "application" [11,p.4]. Roughly, Dxy = z means that formula z is a most general result of applying formula x to formula y, i.e. that z is the result of applying to the consequent of x a most general substitution which transforms the antecedent of x into a substitution instance of y.

A precise definition can be given [5] using the Unification Theorem of J.A. Robinson [9,p.33; 2,p.79]. Although condensed detachment is usually regarded as a method of abridging the presentation of proofs [8,11], Meredith must have been well aware that it could be regarded as a rule of inference which combines the rules of substitution and *modus ponens* into a single rule. In general the rule of condensed detachment is weaker than the joint rules of substitution and *modus ponens*, but every formula derivable from given axioms by means of substitution and *modus ponens* is a substitution instance of a formula which is "D-derivable" from these axioms, i.e. which is derivable from them by using condensed detachment as sole rule of inference [3].

David Meredith [4] recently informed me that in 1957 he programmed UNIVAC to find the D-derivatives of any given set of formulas, but that with the limited capacity of the computer he couldn't get any results which could not have been obtained by hand more quickly. In the sixties more powerful computers became available, and theorem-provers applicable to arbitrary first-order systems were developed. J.A. Robinson's fundamental 1965 paper [9] emphasizes the idea of combining the rules of inference into a single rule or principle, called the "resolution" principle. Because non-classical sentential logics are first-order systems whose terms are the formulas of the logics, resolution-based theorem-provers could be applied to these logics, but I do not know of any studies in this direction. However it is to be expected that a program specifically designed to prove

theorems for non-classical sentential logics will give better results for these systems than a general first-order theorem-proving program, and indeed it is easily shown that condensed detachment can be described as the result of an iteration of Robinson's resolution. Nevertheless, even with a fast modern computer having a large memory, and a program specifically designed for theorem-proving in non-classical sentential logics, the limitations of computer time and space remain a severe constraint, and most of my programming efforts have therefore been devoted to the introduction of heuristics which cause the computer to search more deeply in particular parts of the tree of theorems which are theoretically D-derivable from a given set of axioms.

The program is written in FORTRAN, the formulas in Polish prefix notation being stored in arrays in which the variables are represented by positive integers and the connectives by negative integers. In the rest of this paper it is assumed that all formulas are in Polish notation, so that $A \rightarrow A$ is written Cpp, and the two transitivities are

CCpqCCqrCpr and CCpqCCrpCrq.

By the length of a formula is meant the number of occurrences of variables and connectives in it: thus the two transitivities are each of length 11. The program has been run using three different heuristics, and combinations of these. To explain these heuristics, we need to describe what it means to "develop" a formula. Suppose that the two transitivities, t_1 and t_2, have been stored in the memory of the computer. To develop t_1 we compute Dt_1t_1 and store

it in the memory. Then to develop t_2 we compute Dt_2t_1, Dt_1t_2 and Dt_2t_2 and store them in the memory. Now let x be a formula which has been D-derived from the two transitivities by developing some formula. Then to develop x we compute Dxy and Dyx for all previously developed formulas y, and Dxx, and store each formula z so computed in the memory unless (a) the formula z or some variant of it has already been stored in the memory, or (b) the length of the formula z exceeds some preassigned "length-parameter" ((b) has been called the "length-heuristic" [10,p.11]). Provision is made for varying the length-parameter during the computation ; the maximum value which this parameter takes on affects the number of formulas which can be stored without exceeding the space available in the memory. Our three heuristics can now be described as follows. (1) Breadth-first search, where each time we select a formula x for development we choose the first-computed of the formulas undeveloped at that time. (2) Ordered-list search, where each time we select a formula for development we choose it from among the shortest of the formulas undeveloped at that time. (3) Shortest-consequent search, where each time we select a formula for development we choose it from among the formulas of shortest consequent undeveloped at that time.

The computer was programmed to test whether Cpp and any of the formulas stored had a common substitution instance, and no example was found.

After the program had been run with the ordered-list search, it appeared probable that, including the two transitivities themselves,

exactly 220 theorems of length at most 19 are D-derivable from the two transitivities, and in fact subsequent runs with various combinations of the heuristics and various length-parameters have not succeeded in producing any more D-derivable theorems of length at most 19. In one of these subsequent runs, more than 54,000 combinations Dxy were computed, including all such combinations Dxy with x and y both among the 220 theorems mentioned. Some statistics concerning these 220 theorems follow. Two are of length 11 (the two transitivities), 18 are of length 15, and 200 are of length 19. Among the 18 of length 15, the numbers with consequents of lengths 5,9 and 11 are 2,6 and 10 respectively. Among the 200 of length 19, the numbers with consequents of lengths 5,7,9,11,13 and 15 are 10,8,10,20,60 and 92 respectively. Those with consequents of length 9 (so that antecedent and consequent have the same length) are

CCCCpqCprsCCqtCCtrs

CCCCpqCprsCCstCCqrt

CCCCpqCprsCCtCqrCts

CCCCpqCprsCCtrCCqts

CCCCpqCrqsCCrtCCtps

CCCCpqCrqsCCstCCrpt

CCCCpqCrqsCCtCrpCts

CCCCpqCrqsCCtpCCrts

CCCCpqCrstCCqsCCrpt

CCCCpqCrstCCrpCCqst.

On request I will be pleased to provide interested readers with copies of the list of 220 theorems of length at most 19.

Using the ordered-list search, the computer produced the 220 D-derivable theorems of length at most 19 after developing 32 formulas. With length-parameter 23 this took about 136 seconds, and by this time 600 formulas had been stored; with length-parameter 27 (or larger) about 175 seconds were required, and by this time 720 formulas had been stored - the extra time was probably occupied mainly in testing whether Cpp and the extra 120 formulas of length 27 had a common substitution instance. Using the breadth-first search, with length-parameter changing from 27 to 23 after 600 formulas had been stored, the computer exhausted the available storage space after producing 186 D-derivable theorems of length at most 19 and developing 80 formulas; this took about 7 minutes, and by this time 950 formulas had been stored. Using the shortest-consequent search, with length-parameter changing from 31 to 27, 23 and 19 after 150,250 and 825 formulas had been stored, the computer produced only 155 of the D-derivable theorems of length at most 19 among the first 825 formulas; to produce these 825 formulas took about 6 minutes, and it was necessary to develop 62 formulas; the remaining 65 D-derivable theorems of length at most 19 were produced after a further 4 minutes, and required the development of a further 39 formulas. The program has also been run with the heuristic circulating in various ways among the three different types of search, and produced the 220 D-derivable theorems of length at most 19 more quickly than the shortest-consequent search,

but less quickly than the ordered-list search. The search for more theorems of length at most 19 among the combinations Dxy with x and y in the above set of 220 such theorems took about 17 minutes; naturally it was necessary to develop at least 220 formulas to make this search, and in fact the search required only these 220 to be developed; the program went on running for another 2 minutes and developed another 17 formulas before exhausting the available processing time.

Examination of the computer output suggests various possible properties of the set of formulas D-derivable from the two transitivities. For instance it appears extremely probable that Dxy is defined for any two formulas x and y in this set, i.e. that any two such formulas are applicable to each other. L. Powers [6] recently made a similar observation in a letter to Nuel Belnap, and may since have proved this result [7]. Another interesting question is whether any substitution instance of either of the transitivities is D-derivable from them. None of the D-derivable theorems produced and stored in the computer runs mentioned above has this property.

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A General Gentzen System for Implicational Calculi

I have lost a lot of sleep over P-W recently, and have inflicted the problem on Massey, McCall, and Dwyer. (I have tried to inflict it again on Belnap, but he isn't buying. Asking why he didn't work on it, he replied, "2 years is enough for that problem.") Here, however, is what has been going on lately.

First, Dwyer has applied the Coherence methods to get some nice results. Let

AxB. $B \to C \to . A \to B \to . A \to C$

AxCB. $A \to B \to . B \to C \to . A \to C$

AxI. $A \to A$

P-W is the implicational calculus determined by the B, CB and I axes and modus ponens for $\to$. P-W-I is the calculus determined by B and CB axes alone, with modus ponens. The following are the major points.

Belnap's conjecture. If both $A \to B$ and $B \to A$ are theorems of P-W, then A and B are exactly the same well-formed formula. (Otherwise put: no 2 distinct formulas are provably equivalent in P-W.)

Powers' conjecture. $A \to A$ is not a theorem of P-W-I, for any formula A.

Powers' theorem. Powers' conjecture implies Belnap's conjecture.

Dwyer's results.

(1) Powers' conjecture is strictly equivalent to Belnap's conjecture.

(2) Modus ponens may be replaced in the formulations of P-W and P-W-I with suffixing, prefixing and transitivity rules.

(3) Modus ponens may be replaced in the formulation of P-W with the shuffle rule "From $A \to B$ and $C \to D$ infer $D \to A \to . C \to B$" and the cut rule "From $A \to A \to B$, to infer B".

All of Dwyer's arguments are very simple, including a new proof of Powers' theorem.

I now propose a general Gentzen system for implicational calculi, which, however, I suppose should be credited to Curry. Its present form rests on a suggestion of Belnap's, to introduce the combinators directly into a Gentzen system which I had proposed for P-W and P-W-I.

General Gentzen system.

Formulas built up as usual from → and sentential variables p, q, etc. To avoid confusion with combinators, I henceforth use, x, y, etc. for arbitrary formulas.

Antecedents defined recursively as follows:

(i) B, C, W, K, I are combinators

(ii) If X is a combinator, X is an antecedent

(iii) If X is a formula, X is an antecedent

(iv) If X and Y are antecedents, (XoY) is an antecedent.

Consecution defined as follows:

(v) If X is an antecedent and y is a formula, X ⊢ y is a consecution.

Conventions for formulating general Gentzen system

o is omitted in favor of simple juxtaposition, and association of o is to the left.

...X... is an antecedent with a designated occurence of X, and

...Y... is the result of replacing X with Y in this antecedent.

Rules

B rule. ...(X(YZ))... ⊢ y yields ...(BXYZ)... ⊢ y

C rule. ...(XYZ)... ⊢ y yields ...(CXYZ)... ⊢ y

I rule. ...X... ⊢ y yields ...(IX)... ⊢ y

K rule. ...X... ⊢ y yields ...(KXY)... ⊢ y

W rule. ...(XYY)... ⊢ y yields ...(WXY)... ⊢ y

→* Xy ⊢ z yields X ⊢ y → z

*→ X ⊢ y and ...z... ⊢ w yield ...(y → z X)... ⊢ w

ET X ⊢ y and ...y... ⊢ w yield ...X... ⊢ w

Derived rule

CB rule. ...(Y(XZ))... ⊢ y yields ...(CBXYZ)... ⊢ y

CI rule. ...(XY)... ⊢ z yields ...(CIYX)... ⊢ z

CII rule. ...(XI)... ⊢ z yields ...(CIIX)... ⊢ z

WI rule. ...(XX)... ⊢ z yields ...(WIX)... ⊢ z

KI rule. ...Y... ⊢ z yields ...(KIXY)... ⊢ z

Obviously many associated derived rule along this line may be developed.

Axioms. p ⊢ p, where p is a sentential variable

Note with respect to this Gentzen system: the left side is always an antecedent, and is never empty. The right side is always a formula. Combinators, and the operation o on antecedents, can only occur on the left of a provable (indeed, of a well-formed) consecution.

Note moreover that the usual Gentzen-type principles apply. Any formula, once introduced into a proof, can never again be excised, except by ET. (It may, however, be incorporated as a subformula into longer formulas, by the *→ rules.) Similarly, once a combinator has been introduced, it cannot be excised, either. Note also that there is no general permutation rule within antecedents, or association rule either; a consecution xzy ⊢ w is just different from xzy ⊢ w, x(yz) ⊢ w, etc.

We may vary the general Gentzen system by making a different choice of combinators. For example, we might take CB as a combinator, dropping

C and treating CB as an indissoluble unit governed by the CB-rule, now taken as primitive. So we may think in general of ourselves as dealing with a class of Gentzen systems, differing with respect to the choice of combinators taken as primitive.

We now prove some things about our Gentzen systems. We drop ET as a rule.

Reversibility lemma. Let $X \vdash y \to^* z$ be a theorem of a Gentzen system. Then $Xy \vdash z$ is also a theorem; i.e., the rule $\to^*$ is reversible.

Proof. By induction on the length of proof of $X \vdash y \to^* z$ It is not, at any rate, an axiom. We assume on inductive hypothesis, for all X, y, z, that if $X \vdash y \to^* z$ is provable in k steps, then $Xy \vdash z$ is provable in not more than k steps. We suppose that $X \vdash y \to^* z$ is provable in k+1 steps, and we show that $Xy \vdash z$ is provable in no more steps.

This is evident if $X \vdash y \to^* z$ comes from $Xy \vdash z$ by $\to^*$. Suppose that $X \vdash y \to^* z$ comes from some other 1-premiss rule from $X' \vdash y \to^* z$. On inductive hypothesis, $X'y \vdash z$ is provable. Since all of the rules can still be applied in the presence of an added constituent on the right of the antecedent, except of course $\to^*$, we may derive $Xy \vdash z$ by the same 1-premiss rule in no more steps.

Suppose finally that $X \vdash y \to^* z$ comes by $^*\!\to$. X is then of the form $\ldots((w \to v)X')\ldots$, and $X \vdash y \to^* z$ comes from premisses $X' \vdash w$ and $\ldots v \ldots \vdash y \to^* z$. On inductive hypothesis, $\ldots v \ldots y \vdash z$, whence applying $^*\!\to$, $Xy \vdash z$ is provable in no more steps. This completes the proof of the reversibility lemma.

Proof of ET. Though ET has been dropped as a primitive rule of our Gentzen systems, it is nevertheless admissible in these systems.

Proof. By double induction. We call X ⊢ y the left premiss of ET, ...y... ⊢ w the right premiss, and ...X...⊢ w the conclusion. The primary induction is on the length of y. The secondary induction is on the length of proof of the right premiss; its hypothesis is that, for all left premisses and for all right premisses provable in fewer than k steps, the conclusion of ET is porvable in the Gentzen system; the conclusion of the secondary inudction, which must be demonstrated, is that for all left premisses and for all right premisses provable in k steps, the conclusion of ET is provable in the Gentzen system.

We assume, accordinaly, the hypothesis of the primary induction; that ET can always be performed on formulas shorter than y. And we assume also the hypothesis of the secondary induction. Cases for the secondary induction are as follows:

Case 1. The right premiss is an axiom. Then the left premiss of ET is also its conclusion, and is on assumption provable.

Case 2. The selected occurence of y in the right premiss case comes by a rule which does not newly introduce y. On the inductive hypothesis, we may replace y with X in the premisses of the right premiss, and infer the conclusion of ET by the same rule.

Case 3. y is introduced into the left premiss by the K rule (or the KI rule, with a different choice of primitive combinators). I.e., y occurs as part of Y in the introduced (KZY). Replacing y with X in Y, use the K rule to infer the conclusion of ET.

Case 4. y is introduced quasi-parametrically, in Curry's terminology, by the W (or WI) rule. I.e., y occurs in Y, and the constitutent introduced by the W rule is (WXY). Oh well, I knew that there would be

trouble with W, which--for present purposes-- is hardly worth fixing, since W isn't important for P-W. What has to be done, essentially, is to complicate the argument by allowing simultaneous replacement on inductive hypothesis of the secondary induction on all quasi-parametric ancestors of given Y. I leave this task to the interested reader, noting that multiple replacements of y with X are no more problematic than are single replacement.

Case 5. y is introduced by the rule *→. Then, where y is z→w, the left premiss is X ⊢ z→w, and by the reversibility lemma Xz ⊢ v is provable. The premisses of the right premiss are then ...v... ⊢ w and Y ⊢ z, and the right premiss has the special form ...(z→v Y)... ⊢ w. By the hypothesis of the primary induction on length of formula, ...(Xz)... ⊢ w; and again, ...(XY)... ⊢ w. The latter is the conclusion of ET. This exhausts the cases for the secondary induction, and ends the proof of ET for the general Gentzen systems.

Curry-Meredith-Powers interpretation of Gentzen systems.

We associate with each of the combinators the set of all substitution instances respectively of the following formulas.

With B. $q\to r \to. p\to q \to. p\to r$

With C. $(p \to. q\to r) \to. q \to. p\to r$

With I. $p\to p$

With W. $(p \to. p\to q) \to. (p\to q)$

With K. $p \to. q\to r$

We now define the normal interpretation of an antecedent.

(1) The normal interpretation of a combinator is the set of all substitution instances of the formula associated with that combinator.

(2) The normal interpretation of a formula x is its unit set $\{x\}$.

(3) Suppose that the normal interpretation of an antecedent X is the set X# of formulas, and that the normal interpretation of the antecedent Y is the set Y# of formulas. Then the normal interpretation of the antecedent (XoY) is the set of all formulas B such that, for some formula $A \in Y$, $A \to B$ is in X.

Normal interpretation lemma. Let # be the normal interpretation, and let X, Y, Z be any antecedents. Then the following hold.

For B. $(BXYZ)\# = X(YZ)\#$

For C. $(CXYZ)\# = XZY\#$

For I. $(IX)\# = X\#$

For W. $(WXY)\# \subseteq XYY\#$

For K. $(KXY)\# \subseteq X\#$. If Y# is non-empty, $(KXY)\# = X\#$

We might add, derivatively,

For CB. $(CBXYZ)\# = Y(XZ)\#$

We also note that the set of all substitution instances of the following is associated as its normal interpretation

With CB. $p \to q \to . q \to r \to . p \to r$

Truth on normal interpretation. We now characterize a consecution $X \vdash y$ as true on the normal interpretation just in case $y \in X\#$.

Facts about Truth.

(1) All axioms are true on normal interpretation, since $p \in \{p\}$.

(2) If the premiss of any of the BCI rules is true on normal interpretation, so also is its conclusion. This holds for the K rule too, if the Y introduced by it is such that $Y\# \neq \emptyset$. (Proof by the normal interpretation lemma.)

(3) If the premiss of the rule $\rightarrow^*$ are true on the normal interpretation, so also is its conclusion. (For suppose $z \in Xy\,\#$. By definition, $y \rightarrow z \in X$.)

(4) If the premisses of the rule $*\rightarrow$ are true on the normal interpretation, so also is its conclusion. (For suppose $y \in X\#$ and $w \in \ldots z \ldots \#$. Then, by definition, $(y \rightarrow z\ X)\# = \{z\}$. So, $w \in \ldots (y \rightarrow z\ X) \ldots \#$.)

Normal interpretation theorem. All consecutions provable in the Gentzen system in which neither of the combinators K, W occur are true on the normal interpretation. On a different formulation, this may include primitive combinators CB, CI, CII as well, among the permitted combinators.

Proof. By the facts about truth; the axioms are true, and the rules are truth-preserving, so long as the W and K rules are not used. (The K rule can be used also, whenever the Y introduced by it is such that Y# is guaranteed non-empty; e.g., when Y is a formula or a primitive combinator.)

It should be noted that normal interpretation, though applied here to Gentzen systems, is hardly original with me. Mainly, I owe it to Powers, who credits Dunn with some technical suggestions. (Dunn credits Powers with the main ideas.) Excepting W and K (the latter in part) from the interpretative scheme is implicit in earlier and related work of Curry and Feys and of Prior and Meredith. As noted, I am indebted to Belnap for the suggestion that the combinators themselves should for my purposes be incorporated directly in the Gentzen system.

Since it is P-W with which we are concerned, there is no point about worrying about the exceptions to be made for K and W. Accordingly, a pure combinator is defined for present purposes as follows, recursively.

(i) B, C, and I are pure combinators

(ii) If X and Y are pure combinators, so is XoY.

A P-W combinator is any pure combinator built out of B, CB, and I. (I.e., C only occurs in it in the context (CB).) A P-W-I combinator is any pure combinator built out of B and CB alone.

Completeness theorem for P-W. (Powers). Let x be any formula. Then x is a theorem of P-W iff there is some P-W combinator X such that

(a) $X \vdash x$ is a theorem of the Gentzen system.

Corollary. x is a theorem of P-W iff there is some P-W combinator X such that

(b) $X \vdash x$ is true on the normal interpretation.

Completeness theorem and corollary for P-W-I. Let x be any formula. Then x is a theorem of P-W-I iff there is some P-W-I combinator X such that (a), equivalently (b), holds for X.

Proof. (for P-W, P-W-I being similar). Suppose first that x is a theorem of P-W. If x is a prefixing axiom, prove $B \vdash x$ in the Gentzen system. If x is a suffixing axiom, prove $CB \vdash x$ in the Gentzen system. If x is an identity axiom, prove $X \vdash x$ in the Gentzen system.

Example. $CB \vdash p \to q \to .\ q \to r \to .\ p \to r$. This comes by iterated application of $\to^*$ from $CB(p \to q)(q \to r)p \vdash r$. By the CB rule, this comes from $q \to r(p \to q\ p) \vdash r$. By the $*\to$ rule, this comes from $p \vdash p$ and $q \to r \vdash q \to r$. The former is an axiom. $q \to r \vdash q \to r$ comes by $\to^*$ from $(q \to r)q \vdash r$, which comes by $*\to$ from $q \vdash q$ and $r \vdash r$.

Proof continued. Suppose on inductive hypothesis that for P-W combinators X and Y, and for theorems $x \to y$ and x of P-W, we have $X \vdash x \to y$ and $Y \vdash x$. By the reversibility lemma and ET, $XY \vdash y$. XY is also a P-W term, by definition, so if that if (a) holds for the premisses of an application of modus ponens for $\to$ to get a P-W theorem, it also holds for the

conclusion. Accordingly, (a) holds for all theorems of P-W. By the normal interpretation theorem, (a) implies (b). So if x is a theorem of P-W, there is a P-W combinator X such that both (a) and (b) above are satisfied.

Suppose now that x is not a theorem of P-W. We wish to show that there is no X satisfying (b). But, by the definition of #, it is clear that, for each P-W combinator X, X# is a subset of the theorems of P-W. So x, as a non-theorem, can belong to none of these X#. So if x is a non-theorem of P-W, then for all P-W combinators X, X ⊢ x is false. By the normal interpretation theorem, X ⊢ x is moreover a non-theorem of the Gentzen system. Accordingly, if there is a P-W combinator X satisfying either (a) or (b), x is a theorem of P-W. This completes the proof of the completeness theorems wanted here and their corollaries.

Considerably more could be done with this Gentzenization. E.g., we might allow antecedents to occur on the right as well as one the left. For the present, however, we again stick to what is simple, concentrating on the effort to examine the P-W problem.

We now drop all consideration of K and W, considering only Gentzen systems for B, C, I, CB, etc. Then we can prove

Meredith's lemma. If X ⊢ x is a theorem of the Gentzen system, then there are X', x' such that

(i) X' ⊢ x' is a theorem

(ii) Each sentential variable that occurs in X' ⊢ x' occurs exactly twice in this consecution

(iii) X ⊢ x is a substitution instance of X' ⊢ x', strictly.

Interlude on substitution. A substitution is an endomorphism s on the

algebra of formulas (and antecedents). I.e., a substitution is any function s defined on all formulas and antecedents which satisfies the following conditions:

(iv) $s(x \to y) = s(x) \to s(y)$ for all formulas x, y

(v) $s(X \circ Y) = s(X) \circ s(Y)$

(vi) $s(B) = B$, and similarly for other combinators.

We use E for the set of all substitutions. An antecedent X is a substitution instance of an antecedent X' just in case, for some $s \in E$, $s(X') = X$. Similarly, $X' \vdash x$ just in case there is an $s \in E$ such that $s(X) = X'$ and $s(x) = x'$.

Finally, a substitution is <u>strict</u> provided that, for each sentential variable p, s(p) is also a sentential variable. So a strict substitution rewrites variables as other variables (not necessarily distinct), and may identify some variables. A <u>strict substitution</u> is a <u>permutation</u> provided that it maps the set of sentential variables 1-1 onto itself (and if, for present purposes, only finitely many variables p are such that $s(p) \neq p$).

<u>Proof of Meredith's lemma</u>. Suppose $X \vdash x$ is a theorem of the Gentzen system. We find X', x' and a strict substitution s such that $s(X') = X$, $s(x') = x$, and $X' \vdash x'$ is a theorem of the Gentzen system. (Note: absence of K, W is essential.) If $X \vdash x$ is an axiom $p \vdash p$, let $X' = X$, $x' = x$, and s be the identity function (substitution). If Meredith's lemma is true of the premiss of a 1-premiss rule (excluding K, W), it is clearly true of the conclusion, using the same substitution s. This leaves $\to^*$. We may assume on inductive hypothesis that there are X', y', ...z'..., w' and substitutions s, s' such that (i) these substitutions are

strict, and (ii) $s(X') = X$, $s(y') = y$, $s'(\ldots z' \ldots) = \ldots z \ldots$, and $s'(w') = w$, (iii) $X' \vdash y'$ and $\ldots z' \ldots \vdash w'$. By choosing the 2 distinct statements of (iii) so that they have no vaiables in common (as we may, since the set of theorems of the Gentzen system is easily proved closed under substitution), we may define a new substitution s" which agrees with s on variables of the left member of (iii), and with s' on the right member.

Applying $*\!\!\rightarrow$ to these theorems under (iii) with no variables in common, we may take $\ldots(y' \rightarrow z'\ X') \ldots$ as the antecedent of (i) of Meredith's lemma, w' as its consequent, and s" as the strict substitution guaranteeing (iii). Since each sentential variable occurs exactly twice in this application of $*\!\!\rightarrow$, and since the 2 premisses of $*\!\!\rightarrow$ have been chosen to have no variables in common, each sentential variable occurs exactly twice in the desired provable formula, guaranteeing (ii) of Meredith's lemma as well. This completes the proof of Meredith's lemma.

We are now going to work with the P-W-I combinators, our aim being to show that for no such combinator X is $X \vdash y \rightarrow y$ a Gentzen theorem for any A. By the reversibility lemma, it suffice to show unprovable for all P-W-I combinators S and formulas y all the $Xy \vdash y$. By the P-W-I completeness theorem, this will verify Powers' conjecture; by Powers' theorem, it will moreover verify Belnap's conjecture. As a first installment

Basis lemma. For all sentential variables p and P-W-I combinators X, we have $Xp \vdash p$ as a non-theorem of the Gentzen system.

Proof. No $Xp \vdash p$ are axioms. Nor can any be the conclusion of a rule, since the B and CB rules (the only ones that can be applied to get a

P-W-I term into the antecedent) require at least three distinct components in the antecedent. These extra components can only be introduced by applying $*\rightarrow$, after which the arrows won't go away, so that $Xp \vdash p$ is always underivable under the assumptions of the lemma.

We wish now to apply Meredith's lemma to extend the Basis lemma to the case where an arbitrary formula y takes the place of p in the lemma.

Definition of quasi-identity. A formula $y''\rightarrow y'$ is a quasi-identity provided that, for some strict substitution s, $s(y'') = s(y')$; i.e., provided that $s(y''\rightarrow y')$ is of the form $y\rightarrow y$.

Definition of Meredith quasi-identity. A quasi-identity in which each sentential variable occurs exactly twice is a Meredith quasi-identity.

Quasi-identity lemma. Suppose $Xy \vdash y$, for some P-W-I combinator X. Then $Xy' \vdash y''$, for some Meredith quasi-identity $y'\rightarrow y''$.

Proof by Meredith's lemma.

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